Flow Invariance Preserving Feedback Controllers for the Navier–Stokes Equation

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In this paper we develop a concrete procedure for designing feedback controllers to ensure that the resultant dynamics of turbulence will preserve certain prescribed physical constraints. Examples of such constraints include, in particular, the level sets of well known invariants of the inviscid flow such as helicity. We also bring to light a certain m-accretivity property of suitable quantization of the nonlinearity in the Navier–Stokes equation and utilize the theory of nonlinear semigroups to resolve the controlled Navier–Stokes inclusion with a multi-valued feedback term.

1. INTRODUCTION

Control and management of turbulence dynamics is important in many branches of engineering [23]. Besides traditional fluid dynamic applications,

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fluid type models are also used in characterizing information flow dynamics in communication networks and control [13]. This paper addresses the specific question of finding feedback controllers which ensure that certain desired properties of the state are preserved for the controlled flow. Such constraints usually come from certain engineering specifications or limitations. For example, the requirement can be that of ensuring the enstrophy in a specific spatial region to be kept within a bound. Our feedback controllers will guarantee such a bound on the time evolution of the controlled system. Fluid dynamics in the absence of viscosity has invariants with deep physical significance. One such invariant is the helicity which has been shown to be connected with the knottedness of vortex lines [16]. In Subsection 4.5 we will design controllers which will enforce and maintain a given bound on helicity for the controlled dynamics. In the context of information dynamics and information warfare, control on specific components of flow in specific regions (as in the example in Subsection 4.3) corresponds to instantaneous firewalls.

In Section 2 we describe the governing equations and functional framework. The three main theorems for construction of feedback controllers are stated in Section 3. In Section 4 we formulate five examples for the applications of these theorems. In Section 5 we describe the nonlinear semigroup theory using the theory of m-accretive operators as applied to the Navier–Stokes problem in bounded and unbounded domains in two and three dimensions. This new treatment sets the stage for the mathematical resolution of the Navier–Stokes inclusion problem with the multi-valued feedback term and this is described in Section 6.

The intermediate m-accretive construction of the nonlinearity used in this paper is in fact a form of quantization conceptually similar to constructive quantum field theory [11, 19]. In fact, in the Navier–Stokes equation also, the main mathematical difficulty is due to the fact that in the nonlinearity (inertia term) we are trying to multiply generalized functions. The m-accretive property of the (quantized) nonlinear operators of the Navier–Stokes equation also sets the stage for a number of future research directions in the deterministic as well as stochastic setting [3, 8, 18] exploiting the full power of nonlinear semigroup theory of multi-valued generators and theories on variational and quasi-variational inequalities. In this paper we focus on $L^2$-theory and thus use only Komura type Hilbert space generation theorems. The $L^p$-theory and the full utilization of Crandall–Liggett type generation theorems on Banach spaces might be possible in the future once the $L^p$-accretivity of the Stokes operator (which is currently an open problem) is resolved.
2. CONTROL THEORETIC FORMULATION

Consider the controlled Navier–Stokes equation,

\[
\frac{\partial y(x, t)}{\partial t} + y \cdot \nabla y(x, t) = -\nabla p(x, t) + \nu \Delta y(x, t) + g(x, t) + u(x, t), \quad (x, t) \in Q = \Omega \times (0, T),
\]

(2.1)

\[
\nabla \cdot y(x, t) = 0 \quad \text{in} \quad (x, t) \in Q,
\]

(2.2)

\[
y(x, t) = 0 \quad \text{in} \quad (x, t) \in \Sigma = \partial \Omega \times (0, T),
\]

(2.3)

\[
y(x, 0) = y_0(x), \quad x \in \Omega
\]

in an open and bounded domain \(\Omega \subset \mathbb{R}^n, \quad n = 2, 3\), with a smooth boundary \(\partial \Omega\). We will also provide a treatment of the unbounded exterior domain in this paper. Here \(y = (y_1, y_2, \ldots, y_n)\) is the velocity field, \(p\) is the pressure, \(g = (g_1, g_2, \ldots, g_n)\) is an external force, and \(u = (u_1, u_2, \ldots, u_n)\) is a distributed control on \(\Omega\). As noted in [6, 10, 23] distributed control of fluid flow arises in several applications including electromagnetic (Lorentz force) control of salt water and liquid metals. In information dynamics the type problem control actuation always seems to appear as a distributed term in the state equation with possible spatial localization.

We shall use the standard notations (see, e.g., [25])

\[
H := \left\{ y \in (L^2(\Omega))^n, \nabla \cdot y = 0 \text{ in } D(\Omega), y \cdot n|_{\partial \Omega} = 0 \right\},
\]

(2.5)

\[
V := \left\{ y \in (H^1_0(\Omega))^n, \nabla \cdot y = 0 \text{ in } D(\Omega) \right\},
\]

(2.6)

\[
(Ay, z) := \sum_{i=1}^{n} \int_{\Omega} \nabla y_i \cdot \nabla z_i dx, \quad \text{for all } y, z \in V,
\]

(2.7)

\[
b(y, z, w) := \sum_{i,j=1}^{n} \int_{\Omega} y_i D_i z_j w_j dx, \quad \text{for all } y, z, w \in V.
\]

(2.8)

Let \(B(\cdot) : V \rightarrow V^*\) (the dual space of \(V\)) be defined by

\[
\langle B(y), w \rangle := b(y, y, w), \quad \text{for all } y, w \in V,
\]

(2.9)

and let \(\| \cdot \|\) be the norm of \(H\) (\(L^2\)-norm) and \(\| \cdot \|\) be the norm defined by

\[
\| y \|^2 := \sum_{i=1}^{n} \int_{\Omega} \nabla y_i \cdot \nabla y_i dx.
\]
We note here that for domains bounded in at least one direction, the norm (that is, the usual $H_1$-norm) is equivalent to $\| \cdot \|$. We recall that (see, for example, [15, 25]),

$$b(y, w, z) = -b(y, z, w)$$

and for bounded domains,

$$|b(y, z, w)| \leq C|y|_{s_1}|z|_{s_2}|w|_{s_3},$$

where $| \cdot |_{s_i}$ is the norm of $(H^s(\Omega))^n$ and

$$s_1 + s_2 + s_3 \geq \frac{n}{2}, \quad \text{if } s_i \neq \frac{n}{2} \text{ for all } i = 1, \ldots, n$$

and

$$s_1 + s_2 + s_3 > \frac{n}{2}, \quad \text{if } s_i = \frac{n}{2} \text{ for any of } i = 1, \ldots, n.$$  

In terms of $V, H, A, B(\cdot)$ we can rewrite (2.1) as

$$\frac{dy}{dt} + \nu Ay + B(y) = f(t) + U(t), \quad t \in (0, T),$$

$$y(0) = y_0,$$

where $f = Pg$, $U = Pu$, and $P : (L^2(\Omega))^n \to H$ is the Hodge projection.

Let $K \subset H$ be a closed and convex set such that $0 \in K$. Our concern here is to find a feedback controller $U = \Phi(y)$ such that $y(t) \in K, \forall t \in [0, T]$ if $y_0 \in K$. In other words, one looks for a feedback controller for which the set $K$ is invariant with respect to Navier–Stokes (semi-) flow. This is done by resolving the Navier–Stokes inclusion problem,

$$\frac{dy}{dt} + \nu Ay(t) + B(y(t)) - f(t) + N_K(y(t)) \ni 0, \quad t \in (0, T),$$

where $N_K(y)$ is the Clark normal cone to $K$.

In the examples formulated in Section 4 the above invariance condition corresponds to control with state constraints as studied in [10]. These controllers are constructed in Theorems 3.1 and 3.2 below. In Theorem 3.3 one considers the case where supp$U \subset \omega \times (0, T)$, $\omega$ being a measurable subset of $\Omega$. These theorems are given in Section 3 and proved in Section 6.

The proofs reduce to existence results for multivalued closed loop systems of accretive type associated with Eq. (2.14) and the necessary prerequisites for such a treatment are carried out in Section 5, which has perhaps an interest in itself.

We shall use the standard notations of Sobolev spaces on $\Omega$. Also, we refer to [2, 4, 7] for basic results and notations on nonlinear analysis to be used in the sequel.
3. THE MAIN RESULTS

The first theorem corresponds to the case where the constraint set $K$ is invariant to the operator $(I + \lambda A)^{-1}$. An implication of this condition (see (6.13) in the proof of this theorem in Section 6) expressed in terms of the interior of the normal cone to $K$ will be used in the proof.

**THEOREM 3.1.** Let $K$ be a closed convex subset of $H$ such that $0 \in K$ and

$$
(I + \lambda A)^{-1}K \subset K, \quad \text{for all } \lambda > 0. \quad (3.1)
$$

Let $y_0 \in D(A) \cap K$ and $f \in W^{1,1}([0, T]; H)$. Let $n = 2$. Then there is a feedback controller $U \in L^\infty(0, T; H)$,

$$
U(t) \in -N_K(y(t)), \quad a.e. t \in [0, T), \quad (3.2)
$$

such that the corresponding solution $y(\cdot)$ to the closed loop system (2.14) satisfies

$$
y \in W^{1,\infty}([0, T]; H) \cap L^\infty(0, T; D(A)) \cap C([0, T]; V), \quad (3.3)
$$

$$
\frac{d^+y(t)}{dt} + (-f(t) + \nu Ay(t) + B(y(t)) + N_K(y(t)))^0 = 0, \quad \text{for all } t \in [0, T), \quad (3.4)
$$

$$
y(0) = y_0, \quad (3.5)
$$

$$
y(t) \in K, \quad \text{for all } t \in [0, T]. \quad (3.6)
$$

If $n = 3$ then the above result is local, i.e., it is true on a certain interval $[0, T_0] \subset [0, T]$.

Here

$$
N_K(y) := \{w \in H; (w, y - z) \geq 0, \text{ for all } z \in K\}, \quad (3.7)
$$

is the normal cone to $K$ at $y$ and

$$
y \to (-f + \nu Ay + B(y) + N_K(y))^0 \quad (3.8)
$$

is the minimal section of the multivalued mapping

$$
y \to (-f + \nu Ay + B(y) + N_K(y)). \quad (3.9)
$$

That is, for each $y$, $(-f + \nu Ay + B(y) + N_K(y))^0$ is the projection of the origin on to the closed convex set $(-f + \nu Ay + B(y) + N_K(y))$. Clearly this
mapping is single valued and by (3.4) we see that the feedback controller 
U is given by

\[ U(t) = -f(t) + \nu Ay(t) + B(y(t)) - (-f(t) + \nu Ay(t) + B(y(t)) + N_K(y(t)))^0, \quad \text{for all } t \in [0, T). \tag{3.10} \]

We will see later in specific examples given in Section 4 that the form (3.10) allows us to explicitly construct the feedback control in a way suitable for numerical implementations.

**Theorem 3.2.** Assume that \( n = 2 \) and that \( K \) is a closed convex subset of \( V \) such that \( 0 \in K \). Then for all \( y_0 \in D(A) \cap K \) and \( f \in W^{1,2}([0, T]; H) \) there is a feedback controller \( U \in L^2(0, T; V^*) \) with

\[ U(t) \in -N^*_K(y(t)), \quad \text{a.e. } t \in [0, T) \tag{3.11} \]

such that the corresponding solution

\[ y \in W^{1, \infty}([0, T]; H) \cap W^{1, 2}([0, T]; V) \]

to the closed loop system (2.14) satisfies

\[ \frac{d^* y(t)}{dt} + (-f(t) + \nu Ay(t) + B(y(t)))^0 = 0, \quad \text{for all } t \in [0, T), \tag{3.12} \]

\[ y(0) = y_0, \tag{3.13} \]

\[ y(t) \in K, \quad \text{for all } t \in [0, T]. \tag{3.14} \]

Here

\[ N^*_K(y) := \{ w \in V^*; \langle w, y - z \rangle \geq 0, \text{ for all } z \in K \}, \tag{3.15} \]

is the \( V^* \)-valued normal cone to \( K \) at \( y \) and as in the previous case,

\[ y \rightarrow (-f + \nu Ay + B(y) + N^*_K(y))^0 \tag{3.16} \]

is the minimal section of the multivalued mapping

\[ y \rightarrow (-f + \nu Ay + B(y) + N^*_K(y)). \tag{3.17} \]

This implies as above that

\[ U(t) = -f(t) + \nu Ay(t) + B(y(t)) \tag{3.18} \]

\[ -(-f(t) + \nu Ay(t) + B(y(t)) + N^*_K(y(t)))^0, \quad \text{for all } t \in [0, T). \]

We notice that such a problem was studied earlier by J. L. Lions [15] in a different context.

The next theorem is concerned with the situation where the controller has the support in \( \omega \times [0, T) \). Moreover, the control \( u \) is not divergence free but belongs to \( (L^2(\Omega))^n, n = 2, 3 \).
Theorem 3.3. Let $K_0$ be a closed convex subset of $(L^2(\Omega))^n$, $n = 2, 3$ such that $0 \in K_0$ and

$$P_{K_0}(my) = mP_{K_0}(y), \quad \text{for all } y \in (L^2(\Omega))^n,$$

(3.19)

where $P_{K_0} : (L^2(\Omega))^n \to K_0$ is the projection operator on $K_0$ and $m$ the characteristic function for the measurable set $\omega \subset \Omega$. Let $y_0 \in D(A)$ such that $my_0 \in K_0$ and let $f \in W^{1,2}([0, T]; H)$. Then for each $\lambda > 0$ there is a feedback controller

$$U_\lambda = -\frac{1}{\lambda}(my_\lambda - mP_{K_0}(y_\lambda))$$

(3.20)

such that the solution $y_\lambda$ of the closed loop system (2.14) satisfies

$$y_\lambda \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; D(A))$$

(3.21)

$$\frac{1}{\lambda} \int_0^T d_{K_0}^2(my_\lambda(t))dt \leq C, \quad \text{for all } \lambda > 0.$$

(3.22)

If $n = 3$ this happens on a sufficiently small interval $[0, T_0] \subset [0, T]$. Here $d_{K_0}$ is the distance to the set $K_0$. If $m = 1$ then the above results are true for $K = K_0$ a closed convex subset of $H$ and for

$$U_\lambda = -\frac{1}{\lambda}(y_\lambda - P_K(y_\lambda)).$$

(3.23)

Remark. We can get further insight on the above form of control by noting that, if we define the indicator function,

$$I_K(x) = \begin{cases} 
0 & \text{for } x \in K \\
+\infty & \text{for } x \notin K
\end{cases}$$

we get its subdifferential,

$$\partial I_K(x) = \{ y \in H; (y, x - u) \geq 0, \text{ for all } u \in K \} = N_K(x).$$

Now define the smoothing function,

$$(I_K)_\lambda(x) = \frac{1}{2\lambda}|x - P_K(x)|^2$$

and we get its Gateaux derivative,

$$(\partial I_K)_\lambda(x) = \frac{1}{\lambda}(x - P_K(x)).$$

We note also that the above derivative is also equal to the Yosida approximation of $\partial I_K$

$$(\partial I_K)_\lambda(x) = \frac{1}{\lambda}(x - (I + \lambda \partial I_K)^{-1}(x)), \quad \text{for all } x \in H.$$
4. EXAMPLES

4.1. Enstrophy. Consider the constraint set

\[ K = \{ y \in V; |\nabla \times y| = |\nabla y| = |A^{1/2}y| \leq \rho \}. \] (4.1)

We first note that (3.1) is satisfied in this case. In fact, consider the equation

\[ y + \lambda Ay = f \] (4.2)

taking inner product with \( Ay \)

\[ |A^{1/2}y|^2 + \lambda |Ay|^2 = (f, Ay) \leq |A^{1/2}f||A^{1/2}y| \leq \frac{1}{2}|A^{1/2}f|^2 + \frac{1}{2}|A^{1/2}y|^2. \] (4.3)

Thus, for all \( \lambda > 0 \),

\[ |A^{1/2}y| \leq |A^{1/2}f|, \] (4.4)

which implies \( (I + \lambda A)^{-1}K \subset K \).

We have

\[ N_k(y) = \left\{ w \in H; w = \begin{cases} 0 & \text{if } |\nabla y| < \rho \\ \bigcup_{\lambda > 0} \lambda Ay & \text{if } |\nabla y| = \rho \end{cases} \right\}. \] (4.5)

Then by Theorem 3.1 (see (3.10)) it follows that for the feedback controller

\[ U(t) = \begin{cases} 0 & \text{if } |\nabla y| < \rho \\ Z(y) & \text{if } |\nabla y| = \rho \end{cases} \] (4.6)

with

\[ Z(y) = -\frac{Ay(t)}{|Ay(t)|^2} \left( (f(t), Ay(t)) - \nu|Ay(t)|^2 - b(y(t), y(t), Ay(t)) \right) \]

and the corresponding closed loop system (2.14) with \( y_0 \in D(A) \) and \( f \in W^{1,2}([0, T]; H) \) has a unique solution \( y \in W^{1,\infty}([0, T]; H) \cap L^{\infty}(0, T; D(A)) \) which satisfies

\[ y(t) \in K, \quad \text{for all } t \in [0, T]. \] (4.7)

It is interesting to notice that even in the case of \( n = 3 \) such a result is true on the whole interval \( [0, T] \) because any local solution satisfying the state constraint (4.7) would be global.
4.2. Localized Dissipation. Let
\[
\mathbf{K} = \left\{ y \in \mathbf{V} ; \int_{\omega} |\nabla y(x)|^2 dx \leq \rho^2 \right\},
\]  
where is \( \omega \) an open subset of \( \Omega \) with smooth boundary (of class \( C^1 \)).

Assume that \( n = 2 \). Then by Theorem 3.2, for each \( y_0 \in D(\mathbf{A}) \cap \mathbf{K} \) and \( f \in W^{1,2}([0, T] ; \mathbf{H}) \) the feedback control (3.18) provides a solution
\[
y \in W^{1,\infty}([0, T] ; \mathbf{H}) \cap W^{1,2}([0, T] ; \mathbf{V})
\]
to the corresponding closed loop system which remains in \( \mathbf{K} \), i.e., \( y(t) \in \mathbf{K} \), for all \( t \in [0, T] \). In this case the normal cone \( \mathbf{N}_K(y) \) is given by
\[
\mathbf{N}_K(y) = \left\{ w \in \mathbf{V}^* ; w = \begin{cases} 0 & \text{if } \int_{\omega} |\nabla y(x)|^2 dx < \rho^2 \\ \int_{\omega} |\nabla y(x)|^2 dx = \rho^2 \end{cases} \right\},
\]  
where \( \lambda > 0 \) and \( \varphi'(y) \in \mathbf{V}^* \) is defined by
\[
(\varphi'(y), z) = -2 \int_{\omega} z(x) \cdot \Delta y(x) dx \\
+ 2 \int_{\partial\omega} z \cdot \nabla y \cdot d\sigma, \quad \text{for all } z \in \mathbf{V}.
\]  
Then \( \mathbf{U} \) is given by
\[
\mathbf{U}(t) = \begin{cases} 0 & \text{if } \int_{\omega} |\nabla y(x, t)|^2 dx < \rho^2 \\ \bar{\lambda} \varphi'(y(t)) & \text{if } \int_{\omega} |\nabla y(x, t)|^2 dx = \rho^2, \end{cases}
\]  
where
\[
\bar{\lambda} = \arg \min \left\{ |\nu \mathbf{A} y(t) + \mathbf{B} y(t) - \mathbf{f}(t) + \lambda \varphi'(y(t))|^2 ; \lambda > 0 \right\}.
\]  

4.3. Pointwise Velocity Constraints. Let
\[
\mathbf{K} = \left\{ y = (y_1, y_2) \in \mathbf{V} ; a_i \leq y_i \leq b_i, i = 1, 2 \right\},
\]  
where \( a_i \leq 0 \leq b_i, i = 1, 2 \). We may apply Theorem 3.2 to get a feedback controller \( \mathbf{U}(t) \in L^2(0, T ; \mathbf{V}^*) \) for which the solution \( y \) to (2.14) remains in \( \mathbf{K} \) if \( y_0 \) does. By (3.18) we see that
\[
\mathbf{U}(x, t) = \begin{cases} 0 & \text{in } \{(x, t); a_1 < y_1(x, t) < b_1; a_2 < y_2(x, t) < b_2\} \cup \mathcal{E} \\ (y_1(x, t), 0) & \text{in } \{(x, t); a_1 < y_1(x, t) < b_1; y_2(x, t) = a_2, b_2\} \\ (0, y_2(x, t)) & \text{in } \{(x, t); y_1(x, t) = a_1, b_1; a_2 < y_2(x, t) < b_2\}, \end{cases}
\]  
where
\[
\mathcal{E} = \left\{ (x, t) \in \mathcal{Q}; y_1(x, t) = a_1, y_2(x, t) = a_2 \right\} \\
\cup \left\{ (x, t) \in \mathcal{Q}; y_1(x, t) = a_1, y_2(x, t) = b_2 \right\} \\
\cup \left\{ (x, t) \in \mathcal{Q}; y_1(x, t) = b_1, y_2(x, t) = a_2 \right\} \\
\cup \left\{ (x, t) \in \mathcal{Q}; y_1(x, t) = b_1, y_2(x, t) = b_2 \right\}.
\]
Applying Theorem 3.3 we may also treat the case of local constraints, 

\[ \mathbf{K} = \left\{ \mathbf{y} = (y_1, y_2) \in (L^2(\Omega))^n, a_i \leq y_i |_{\omega} \leq b_i, i = 1, 2 \right\} \]

where \( \omega \) is a measurable subset of \( \Omega \).

4.4. Pointwise Vorticity Constraint. Theorems 3.2 and 3.3 apply to the case of pointwise vorticity constraint defined as

\[ \mathbf{K} = \left\{ \mathbf{y} \in \mathbf{V} : |(\nabla \times \mathbf{y})(x)| \leq \rho, \text{ a.e } x \in \Omega \right\}. \tag{4.14} \]

It is easily seen that

\[ \mathbf{N}_\mathbf{K}(\mathbf{y}) = \left\{ \mathbf{w} \in \mathbf{V}^* : \mathbf{w} = \nabla \times \mathbf{\mu}, \mathbf{\mu} \in (L^2(\Omega))^n \right\}, \tag{4.15} \]

where

\[ \mathbf{\mu}(x) = \begin{cases} 0 & \text{a.e in } [x \in \Omega; |\nabla \times \mathbf{y}(x)| < \rho] \\ \bigcup_{\lambda > 0} \lambda \nabla \times \mathbf{y}(x) & \text{a.e in } [x \in \Omega; |\nabla \times \mathbf{y}(x)| = \rho] \end{cases} \tag{4.16} \]

and therefore we get

\[ \mathbf{U}(t)(x) = \begin{cases} 0 & \text{a.e in } [(x,t); |\nabla \times \mathbf{y}(x,t)| < \rho] \\ \left( \frac{\lambda \nabla \times \mathbf{y}(x,t)}{\|\nabla \times \mathbf{y}(x,t)\|} \right) \nabla \times \mathbf{y}(x,t) & \text{a.e in } [(x,t); |\nabla \times \mathbf{y}(x,t)| = \rho]. \end{cases} \]

4.5. Helicity Invariance. Let

\[ \mathbf{D} = \left\{ \mathbf{y} \in \mathbf{V} : \left| \int_\Omega \mathbf{y}(x) \cdot (\nabla \times \mathbf{y}(x)) dx \right| \leq \rho^2 \right\}. \tag{4.17} \]

Let us denote the helicity of \( \mathbf{y} \) by

\[ H(\mathbf{y}) = \int_\Omega \mathbf{y}(x) \cdot (\nabla \times \mathbf{y}(x)) dx. \tag{4.18} \]

It is readily seen that the function \( \Phi(\mathbf{y}) := H(\mathbf{y})^2 \) is continuous in \( \mathbf{V} \) and so \( \mathbf{D} \) is a closed subset of \( \mathbf{V} \). Let \( \mathbf{K} \) be the closed convex hull of \( \mathbf{D} \) in \( \mathbf{H} \). Then Theorem 3.3 applies to the present situation and yields a sequence of feedback controllers \( \mathbf{U}_\lambda \in L^2(0, T; \mathbf{H}) \),

\[ \mathbf{U}_\lambda(t) = \frac{1}{\lambda} (\mathbf{y}_\lambda(t) - \mathbf{P}_\mathbf{K}(\mathbf{y}_\lambda(t))), \quad \text{for all } t \in [0, T_0] \tag{4.19} \]

such that

\[ \lim_{\lambda \to 0} \int_0^T d^2_\mathbf{K}(\mathbf{y}_\lambda(t)) dt = 0. \tag{4.20} \]

The projection \( \mathbf{P}_\mathbf{K} \) on \( \mathbf{K} \) is hard to compute so for practical purposes, we shall replace it by the projection on \( \mathbf{D} \).
In this case, \( P_0(y) = z \) is the solution to the problem
\[
\begin{align*}
    z + 4\lambda H(z) \nabla \times z &= y & \text{in } \Omega, \\
    \nabla \cdot z &= 0 & \text{in } \bar{\Omega}, \\
    z &= 0 & \text{in } \partial \Omega,
\end{align*}
\]
with \( \lambda > 0 \). Hence we shall replace \( U_\lambda \) by
\[
\tilde{U}_\lambda = 4H(z) \nabla \times z, \tag{4.22}
\]
where \( z \) is the solution to (4.21).

5. A NONLINEAR SEMIGROUP APPROACH TO THE NAVIER–STOKES EQUATION

A convenient way to treat the multivalued closed loop systems arising in the above theorems is to use the machinery of nonlinear differential equations of accretive type in Banach spaces. Besides the present interest, this seems to be an easy way to treat the Navier–Stokes equations in 2-D and 3-D. This approach has been used previously for the 2-D case in [5].

Define the modified (quantized) nonlinearity
\[
B_N(y) \colon V \to V^*,
\]

\[
B_N(y) := \begin{cases} 
    B(y) & \text{if } \|y\| \leq N \\
    \left( \frac{N}{\|y\|} \right)^2 B(y) & \text{if } \|y\| > N.
\end{cases} \tag{5.1}
\]

By (2.11) we have for \( n=3 \) and for the case of \( \|y\|, \|z\| \leq N \),
\[
|B_N(y) - B_N(z), y - z| = |b(y - z, y - z, y)| \\
\leq C\|y - z\|^{3/2}\|y - z\|^{1/2}\|y\| \\
\leq C\|y - z\|^2 + \frac{\nu}{2} \|y - z\|^2. \tag{5.2}
\]

Similarly for the case of \( \|y\|, \|z\| > N \) we have
\[
|B_N(y) - B_N(z), y - z| \leq C\|y - z\|^2 + CN^2\|y - z\| \\
\leq C\|y - z\|^2 + \frac{\nu}{2} \|y - z\|^2. \tag{5.3}
\]

Similar estimates are obtained for the cases \( \|y\| > N, \|z\| \leq N, \) and \( \|y\| \leq N, \|z\| > N \) (more details will be shown in Subsection 5.1 in the context of
3-D exterior domains). For the 2-D case the final estimate is the same (we notice that the situation is exactly the same for unbounded domain and we will see it later in Subsection 5.1). We thus combine the above cases to conclude that

\[
|(B_N(y) - B_N(z), y - z)| \leq C_N|y - z|^2
+ \frac{\nu}{2}\|y - z\|^2, \quad \text{for all } y, z \in V. \tag{5.4}
\]

Consider the operator \(\Gamma_N : D(\Gamma_N) \to H\) defined by

\[
\Gamma_N = \nu A + B_N, \quad D(\Gamma_N) = D(A). \tag{5.5}
\]

By (2.11) we see that

\[
|B_N(y)| \leq C|Ay|^{3/4}\|y\|^{1/4}
\leq CN^{3/4}|Ay|^{3/4}, \quad \text{for all } y \in D(A), \tag{5.6}
\]

and hence \(\Gamma_N\) is well defined in \(H\).

**Lemma 5.1.** There exists \(\alpha_N > 0\) such that \(\Gamma_N + \alpha_N I\) is m-accretive (maximal monotone) in \(H \times H\).

**Proof.** By (5.4) we see that

\[
((\Gamma_N + \lambda)y - (\Gamma_N + \lambda)z, y - z) \geq \frac{\nu}{2}\|y - z\|^2, \quad \text{for all } y, z \in D(\Gamma_N) \tag{5.7}
\]

for \(\lambda \geq C_N\).

Next we consider the operator

\[
F_N(y) = \nu Ay + B_N(y) + \alpha_N y, \quad \text{for all } y \in D(F_N), \tag{5.8}
\]

with

\[
D(F_N) = \{ y \in V; \nu Ay + B_N(y) \in H \}, \tag{5.9}
\]

where \(\alpha_N \geq C_N\) will be precisely characterized later. Since \(F_N\) is monotone, continuous, and coercive from \(V\) to \(V^*\), it is maximal monotone in \(H\) with domain \(D(F_N) \supseteq D(A)\) (see [7, Chap. II, Example 2.3.7]). We shall prove that, in fact, \(F_N = \Gamma_N + \alpha_N I\) is, for \(\alpha_N\) sufficiently large, m-accretive with domain \(D(F_N) = D(A)\).

We note that by (5.6) we have

\[
|B_N(y)| \leq \delta|Ay| + C_N^{3/4}, \quad \text{for all } y \in D(A) \tag{5.10}
\]

for all \(\delta > 0\). This yields

\[
|Ay| \leq C_N^{3/4}(|F_N(y)| + 1), \quad \text{for all } y \in D(A). \tag{5.11}
\]
We shall denote by $C_N^i$ several positive constants which are independent of $y$.

Next we consider operators
\begin{align}
F_N^1 &= \nu(1 - \varepsilon)A, \quad D(F_N^1) = D(A), \quad (5.12) \\
F_N^2 &= \varepsilon vA + B_N(\cdot) + \alpha_N I, \quad D(F_N^2) = \{y \in V; F_N^2(y) \in H\}, \quad (5.13)
\end{align}

where (for example, $\varepsilon = 1/4$) $\alpha_N$ is large enough such that $F_N^2$ is maximal monotone in $H \times H$. As seen above this happens if $\alpha_N \geq C_N/\nu \varepsilon$. By (5.10) we have
\begin{align}
|F_N^2(y)| &\leq \frac{\nu}{4}|Ay| + |B_N(y)| + \alpha_N|y| \\
&\leq \left(\frac{\nu}{4} + \delta\right)|Ay| + C_N^0 + \alpha_N|y| \\
&\leq \rho|F_N^2(y)| + \alpha_N|y| + C_N^1, \quad \text{for all } y \in D(F_N^1) = D(A),
\end{align}

where $0 < \rho < 1$.

Then by a well known perturbation theorem for nonlinear $m$-accretive operators (see, e.g., [2, Theorem 3.5, Chap. III]) it follows that $F_N^1 + F_N^2$ with domain $D(A)$ is $m$-accretive in $H$. Since $F_N^1 + F_N^2 = \Gamma_N + \alpha_N I$ we infer that $\Gamma_N + \alpha_N I$ is $m$-accretive as claimed.

An immediate consequence of Lemma 1 (see, e.g., [2, 4]) is that for $y_0 \in D(A)$ and $f \in W^{1,1}([0, T]; H)$ the Cauchy problem
\begin{align}
\frac{dy}{dt} + \nu Ay(t) + B_N(y(t)) &= f(t), \quad \text{a.e. } t \in (0, T), \quad (5.14) \\
y(0) &= y_0, \quad (5.15)
\end{align}

has a unique solution $y \in W^{1,\infty}([0, T]; H) \cap L^{\infty}(0, T; D(A))$. Moreover,
\begin{align}
\frac{d^+y}{dt}
\end{align}

exists everywhere on $[0, T)$ and (5.14) is satisfied with $d^+/dt$ instead $d/dt$ everywhere on $[0, T)$. Moreover, $y(t)$ is the limit in $H$ of the discrete approximation scheme,
\begin{align}
\frac{y_{i+1} - y_i}{h} + \nu Ay_{i+1} + B_N(y_{i+1}) = f_i, \quad i = 0, \ldots, N, \quad (5.16)
\end{align}

where $Nh = T$ and
\begin{align}
f_i = \frac{1}{h} \int_{i\Delta t}^{(i+1)\Delta t} f(t)dt. \quad (5.17)
\end{align}

For $y_0 \in H$ and $f \in L^1(0, T; H)$, Eq. (5.14) has a unique “mild solution” which is a limit in $C([0, T]; H)$ of a sequence of strong solutions.
It turns out that for \( N \) large enough, \( y_N \) is the solution to Navier–Stokes equation,

\[
\frac{dy}{dt} + \nu Ay(t) + B(y(t)) = f(t),
\]

\( y(0) = y_0, \)

on a certain interval \((0, T_0)\) where \( T = T_0 \) if \( n = 2 \). The arguments are as follows. Multiplying Eq. (5.14) by \( y_N \) and \( Ay_N \) respectively we get as usual the estimates

\[
|y_N(t)|^2 + 2\nu \int_0^t |y_N(s)|^2 ds \leq |y_0|^2 + \int_0^t |f(s)|^2 ds,
\]

\( n = 3, \) and for \( n = 2 \) the second estimate is improved as

\[
\|y_N(t)\|^2 + \nu \int_0^t |Ay_N(s)|^2 ds \leq \|y_0\|^2 + C \int_0^t |Ay_N(s)|^{3/2} \|y_N(s)\|^{3/2} ds
\]

\[
+ \left( \int_0^t |f(s)|^2 ds \right)^{1/2} \left( \int_0^t |Ay_N(s)|^2 ds \right)^{1/2}.
\]

This implies

\[
\|y_N(t)\| \leq C, \quad \text{for all } t \in [0, T]
\]

for \( n = 2 \) and

\[
\|y_N(t)\| \leq C, \quad \text{for } t \in [0, T_0] \subset [0, T]
\]

if \( n = 3 \) where \( C \) is independent of \( N \). Hence for \( N \geq C, \ B_N(y_N) = B(y_N) \) and, hence \( y_N = y \) is a solution to Eq. (5.18) on \([0, T]\) if \( n = 2 \) and on some interval \([0, T_0]\) if \( n = 3 \). In this way one may find the standard existence results for the Navier–Stokes in 2-D and 3-D. It is useful to notice that since by (5.19)

\[
\text{meas}\{t \in [0, T]; \|y_N(t)\| > N\} \leq \frac{C}{\sqrt{N}}
\]

we conclude that for each \( \varepsilon > 0 \), there is \( y_{\varepsilon} \in W^{1,\infty}([0, T]; H) \cap L^\infty(0, T; D(A)) \) which satisfies Eq. (5.18) on \([0, T]\) \( \setminus E_\varepsilon \) where \( \text{meas}(E_\varepsilon) \leq \varepsilon \) (here \( \text{meas}(\cdot) \) is the Lebesgue measure).
Moreover, it follows by (5.16) and the discrete analogs of the estimates (5.19) and (5.20) that the discrete scheme

$$\frac{y_{i+1} - y_i}{h} + \nu Ay_{i+1} + B_N(y_{i+1}) = f, \quad i = 0, \ldots, N, Nh = T \quad (5.25)$$

is convergent to the solution $y_N$ to (5.14). Here, in order to obtain the discrete estimates we take the inner product respectively with $y_k$ and $Ay_k$ and perform estimates (details are omitted here) very similar to the continuous case. Finally, by estimate (5.19) it follows that for $N \to \infty$, $\{y_N\}$ is weakly convergent in $L^2(0, T; V^*)$ to a weak solution to the Navier–Stokes equation (5.18).

5.1. Exterior Hydrodynamics. In this subsection we will consider three dimensional exterior hydrodynamics and show that in this case too the type of quantization introduced in this paper results in m-accretivity and hence the theory of accretive operators and nonlinear semigroups described in the previous section applies. We start with the stationary problem in an exterior domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial \Omega$ and a smooth boundary data $y^*$,

$$-\Delta y_e + y_e \cdot \nabla y_e + \nabla q = f_e \quad \text{in} \ \Omega, \quad (5.26)$$

$$\nabla \cdot y_e = 0 \quad \text{in} \ \Omega, \quad (5.27)$$

$$y_e|_{\partial \Omega} = y^*, \quad (5.28)$$

$$y_e \to 0 \quad \text{as} \ |x| \to \infty. \quad (5.29)$$

It is well known that a smooth solution of this problem exists for all Reynolds numbers [14] (is unique for low Reynolds numbers [1]) and they exhibit the decay properties [17]

$$|y_e| \leq \frac{C_1}{(1 + |x|)}, \quad \text{and} \quad |\nabla y_e| \leq \frac{C_2}{(1 + |x|^2)}. \quad (5.30)$$

We note that these estimates only concern the 3-D exterior domain. The stationary problem of 2-D exterior domains involves issues concerning the Stokes paradox and are not completely resolved. Consider the controlled time dependent 3-D exterior problem (in the case of 2-D we can consider the same problem with the special case of $y_e = 0$)

$$\partial_t y_T(x, t) + y_T \cdot \nabla y_T(x, t)$$

$$= -\nabla p_T(x, t) + \nu \Delta y_T(x, t) + g_T(x, t)$$

$$+ u(x, t), (x, t) \in Q = \Omega \times (0, T), \quad (5.31)$$
\[ \nabla \cdot y_f(x, t) = 0 \quad \text{in } (x, t) \in Q, \quad (5.32) \]
\[ y_f(x, t) = y^*(x) \quad \text{in } (x, t) \in \Sigma = \partial \Omega \times (0, T), \quad (5.33) \]
\[ y_f(x, 0) = y_{0f}(x), \quad x \in \Omega, \quad (5.34) \]
\[ y_f \to 0 \quad \text{as } |x| \to \infty. \quad (5.35) \]

We consider the nonlinear stability problem as a perturbation of the stationary problem above. Thus set \( y_f(x, t) = y_e(x) + y(x, t) \) and \( p_f(x, t) = q(x) + p(x, t) \) so that \( (y, p) \) satisfies

\[ \partial_t y(x, t) + y \cdot \nabla y(x, t) + y_e \cdot \nabla y(x, t) + y \cdot \nabla y_e(x, t) \]
\[ = -\nabla p(x, t) + \nu \Delta y(x, t) + g(x, t) \]
\[ + u(x, t), (x, t) \in Q = \Omega \times (0, T), \quad (5.36) \]
\[ \nabla \cdot y(x, t) = 0 \quad \text{in } (x, t) \in Q, \quad (5.37) \]
\[ y(x, t) = 0 \quad \text{in } (x, t) \in \Sigma = \partial \Omega \times (0, T), \quad (5.38) \]
\[ y(x, 0) = y_0(x), x \in \Omega, \quad (5.39) \]
\[ y \to 0 \quad \text{as } |x| \to \infty. \quad (5.40) \]

This will give us the abstract equation

\[ \frac{dy}{dt} + \nu A y(t) + B(y(t)) + L_e y = f(t) + U(t), t \in (0, T), \quad (5.41) \]

with

\[ y(0) = y_0, \quad (5.42) \]

where the additional term is \( L_e y = P(y_e \cdot \nabla y + y \cdot \nabla y_e) \). We will now work with the special characteristics of the Sobolev embedding theorems in three dimensional exterior domains and also the regularity and decay properties of the stationary solution \( y_e \) in such domains to establish the accretivity and nonlinear semigroup characterization for this situation. The nonlinearity is truncated in this case in the following way. Let us denote \( \| \cdot \|_1 := \| \cdot \|_{H^1(\Omega)} \). We will truncate using this norm instead of the norm \( \| \cdot \| \) in the previous cases. This is because in the (three dimensional) exterior domain the norm \( \| \cdot \| \) only dominates the \( L^6 \)-norm and not the \( L^2 \)-norm. We thus define

\[ B_N(y) := \begin{cases} 
B(y) & \text{if } \|y\|_1 \leq N \\
\left( \frac{N}{\|y\|_1} \right)^2 B(y) & \text{if } \|y\|_1 > N.
\end{cases} \quad (5.43) \]

We will now prove the \( m \)-accretivity of \( \nu A + B_N \). Let us first estimate

\[ (B_N(y) - B_N(z), y - z). \]
Thus, we give more details in this case. For \( \|y\|_1, \|z\|_1 \leq N \), we have
\[
(B_N(y) - B_N(z), y - z) = (B(y) - B(z), y - z) = b(y - z, y, y - z).
\]

Thus,
\[
\|B_N(y) - B_N(z), y - z)\| \leq \|y - z\|_4^2 \|y\| \leq C\|y - z\|^{1/2}\|y - z\|^{3/2}
\leq \epsilon\|y - z\|_1^2 + C\|y - z\|_3^2.
\]

We will now consider the case of\( B_N(y) - B_N(z), y - z) \)
\[
= \frac{N^2}{\|y\|_1^2} (B(y) - B(z), y - z)
\]
\[
+ \left( \frac{N^2}{\|y\|_1^2} - \frac{N^2}{\|z\|_1^2} \right) (B(z), y - z).
\]
\[
= \frac{N^2}{\|y\|_1^2} b(y - z, y, y - z)
\]
\[
+ N^2 \left( \frac{\|z\|_1^2 - \|y\|_1^2}{\|y\|_1^2 \|z\|_1^2} \right) b(z, z, y - z).
\]

Thus,
\[
\|B_N(y) - B_N(z), y - z)\| \leq C\|y - z\|_4^2 + \frac{N^2\|y - z\|_1}{\|y\|_1 \|z\|_1} \|b(z, z, y - z)\|
\leq \left( \frac{N^2\|y - z\|_1}{\|y\|_1 \|z\|_1} \right) \|b(z, z, y - z)\|
\leq C\|y - z\|^{1/2}\|y - z\|_3
\leq \epsilon\|y - z\|_1^2 + C\|y - z\|_3^2.
\]

We now consider the case \( \|y\|_1 > N, \|z\|_1 \leq N \),
\[
(B_N(y) - B_N(z), y - z) = \frac{N^2}{\|y\|_1^2} (B(y), y - z) - (B(z), y - z)
\]
\[
= \frac{N^2}{\|y\|_1^2} (B(y) - B(z), y - z)
\]
\[
+ \left( \frac{N^2}{\|y\|_1^2} - 1 \right) (B(z), y - z).
\]
which is deduced from the estimate above, and

\[ b(y - z, y, y - z) + \left( \frac{N^2 - \|y\|_1^2}{\|y\|_1^2} \right) b(z, y - z). \]

Thus, noting that \( \|y\|_1^2 - N^2 \leq \|y\|_1^2 - \|z\|_1^2 \), we estimate (in the second trilinear term apply Young’s inequality with 6-2-3),

\[ |(B_N(y) - B_N(z), y - z)| \leq C|y - z|_1^2 + C\|y - z\|_3|y - z|_3 \]
\[ \leq C\|y - z\|_1^{3/2}\|y - z\|^{1/2} \]
\[ \leq \epsilon\|y - z\|_1^2 + C\|y - z\|^2. \]  

(5.45)

We now estimate

\[ |(L_e(y) - L_e(z), y - z)| \leq |b(y - z, y_e, y - z)| \leq |\nabla y_e|_\infty|y - z|^2 \]
\[ \leq C|y - z|^2. \]

Other estimates used earlier in Section 5 (and later in Section 6) are

\[ |(L_e(y), y)| \leq C|y|^2 \]
which is deduced from the estimate above, and

\[ |(L_e(y), Ay)| \leq |b(y, y_e, Ay)| + |b(y_e, y, Ay)| \]
\[ \leq |y|\|\nabla y_e|_A|y_e| + |y_e|\|\nabla y|Ay| \leq \epsilon|Ay|^2 + C(y)^2 + \|y\|^2. \]

We will also estimate \( |B_N(y)| \) and \( |L_e(y)| \). Consider

\[ (L_e(y), z) = b(y, y_e, z) + b(y_e, y, z). \]  

(5.46)

Thus

\[ |(L_e(y), z)| \leq C(|y| + |y_e|)|z| \]

(5.47)

and hence

\[ |L_e(y)| \leq C\|y\|_1. \]  

(5.48)

Similarly

\[ (B_N(y), z) = \begin{cases} 
    b(y, y, z) & \text{if } \|y\|_1 \leq N \\
    \frac{N}{\|y\|_1^2} b(y, y, z) & \text{if } \|y\|_1 > N.
\end{cases} \]  

(5.49)

Hence, for \( n=2 \),

\[ |(B_N(y), z)| \leq |y|_4|\nabla y|_4|z| \]

(5.50)
which gives immediately (see the proof of Theorem 4.1 in [22])

$$|(\mathbf{B}_N(y), z)| \leq |y|^{1/2} \|y\|^{1/2} \left( \|y\|^{1/2} |\mathbf{A}y|^{1/2} + \|y\| \right) |z|.$$ 

Thus,

$$|\mathbf{B}_N(y)| \leq |y|^{1/2} \|y\|^{1/2} \left( \|y\|^{1/2} |\mathbf{A}y|^{1/2} + \|y\| \right).$$

Similarly for 3-D exterior domains,

$$|(\mathbf{B}_N(y), z)| \leq |y|_0 \|\nabla y\|_3 |z|. \quad (5.51)$$

Thus, using the estimate for $|\nabla y|_3$ for 3-D exterior hydrodynamics [12, Lemma 1, Estimate (6)] we get

$$|(\mathbf{B}_N(y), z)| \leq |y|_0 \left( \|y\|^{1/2} |\mathbf{A}y|^{1/2} + \|y\| \right) |z|. \quad (5.52)$$

Thus

$$|\mathbf{B}_N(y)| \leq \|y\| \left( \|y\|^{1/2} |\mathbf{A}y|^{1/2} + \|y\| \right). \quad (5.53)$$

We will now estimate in 2-D,

$$|(\mathbf{B}_N(y), \mathbf{A}y)| \leq |y|^{1/2} \|y\|^{1/2} \left( \|y\|^{1/2} |\mathbf{A}y|^{1/2} + \|y\| \right) |\mathbf{A}y| \quad (5.54)$$

which gives

$$|(\mathbf{B}_N(y), \mathbf{A}y)| \leq \epsilon |\mathbf{A}y|^2 + C \|y\|^2 \left( |y|^2 \|y\|^2 + \|y\| |y| \right). \quad (5.55)$$

Similarly for 3-D

$$|(\mathbf{B}_N(y), \mathbf{A}y)| \leq |y|_0 \left( \|y\|^{1/2} |\mathbf{A}y|^{1/2} + \|y\| \right) |\mathbf{A}y| \quad (5.56)$$

which gives

$$|(\mathbf{B}_N(y), \mathbf{A}y)| \leq \epsilon |\mathbf{A}y|^2 + C \|y\|^2 \left( |y|^4 + \|y\|^2 \right). \quad (5.57)$$
6. PROOFS OF THE MAIN RESULT

6.1. Proof of Theorem 3.1. We will prove that the operator
\[ y \rightarrow \nu Ay + B_N(y) + N_K(y) + \alpha_N y, \]
where \( \alpha_N \) is as in Lemma 1, is \( m \)-accretive in \( H \times H \). Since the accretivity is obvious (this follows from the earlier developments and the fact that the normal cone is maximal monotone) it remains to show the range condition \( R(\mu I + \nu A + B_N(\cdot) + N_K(\cdot)) = H \) for \( \mu > \alpha_N \). Consider the Yosida approximation
\[ F_\lambda = \frac{1}{\lambda} (I - (I + \lambda N_K)^{-1}) \] of \( N_K \).

Let \( f \in H \) be arbitrary but fixed. Then the equation
\[ \nu Ay + B_N(y) + F_\lambda(y) + \mu y = f \] (6.1)
has a unique solution \( y_\lambda \in D(A) \). In fact, to see the uniqueness we set \( z_\lambda = y_\lambda - w_\lambda \), where \( y_\lambda \) and \( w_\lambda \) are two solutions for the same \( f \). Then,
\[ \nu A z_\lambda + (B_N(y_\lambda) - B_N(w_\lambda)) + (F_\lambda(y_\lambda) - F_\lambda(w_\lambda)) + \mu z_\lambda = 0. \]

Taking the inner product with \( z_\lambda \) we get
\[ \nu \|z_\lambda\|^2 + (B_N(y_\lambda) - B_N(w_\lambda), z_\lambda) + (F_\lambda(y_\lambda) - F_\lambda(w_\lambda), z_\lambda) + \mu \|z_\lambda\|^2 = 0. \]

Using (5.4) and the Lipschitz property of \( F_\lambda \) we get
\[ \nu \|z_\lambda\|^2 - \epsilon \|z_\lambda\|^2 - C\|z_\lambda\|^2 - C\|z_\lambda\|^2 + \mu \|z_\lambda\|^2 \leq 0. \]

Hence, we conclude (by taking \( \epsilon < \nu \), and \( (C + C_\nu) < \mu \)) that \( z_\lambda = 0 \).

Taking the inner product of (6.1) by \( y_\lambda \) and \( Ay_\lambda \) respectively we get
\[ \mu \|y_\lambda\|^2 + \nu \|y_\lambda\|^2 \leq |f| \|y_\lambda\|, \] (6.2)
\[ \mu \|y_\lambda\|^2 + \nu |Ay_\lambda|^2 \leq |Ay_\lambda|^3/2 \|y_\lambda\|^3/2 + |f| \|Ay_\lambda\| \] (6.3)
because by condition (3.1) and Proposition 1.1, part (iv) of [2, Chap. IV.1.3]
\[ (Ay_\lambda, F_\lambda y_\lambda) \geq 0 \quad \text{and} \quad 0 \in K. \]

This yields
\[ \|y_\lambda\|^2 + |Ay_\lambda|^2 \leq C, \quad \text{for all} \ \lambda > 0, \] (6.4)
and consequently (see (5.6)),
\[ |B_N(y_\lambda)| \leq CN^{3/4} |Ay_\lambda|^{3/4} \leq C_N \] (6.5)
\[ |F_\lambda y_\lambda| \leq C_N, \] (6.6)
where $C_N$ is independent of $\lambda$. Thus on a subsequence, again denoted $\lambda$, we have

\begin{align}
y_\lambda &\to y &\text{strongly in } V, &\quad (6.7) \\
A y_\lambda &\to Ay &\text{weakly in } H, &\quad (6.8) \\
F_n y_\lambda &\to \eta &\text{weakly in } H, &\quad (6.9) \\
B_N(y_\lambda) &\to B_N(y) &\text{weakly in } H \text{ and strongly in } V^*. &\quad (6.10)
\end{align}

It is readily seen that $\eta \in N_K(y)$ and so $y$ is the solution to

$$
\nu Ay + B_N(y) + \mu y + N_K(y) \ni f.
$$

Let

$$
y \in D(A) \cap K = D(\nu A + B_N(\cdot) + N(\cdot))
$$

and $f \in W^{1,1}([0, T]; H)$. Then the Cauchy problem

$$
\frac{dy(t)}{dt} + \nu Ay(t) + B_N(y(t)) + N_K(y(t)) \ni f(t), \quad \text{a.e. } t \in [0, T]
$$

has a unique solution $y_N \in W^{1,\infty}([0, T]; H)$ which satisfies

$$
\frac{d^2 y_N(t)}{dt^2} + (\nu Ay_N(t) + B_N(y_N(t)) + N_K(y_N(t)) - f(t))^0 = 0, \quad \text{for all } t \in [0, T].
$$

Moreover, by (3.1) and (5.1), respectively. By (6.13) and (6.14) we know that

$$
|\langle Ay_N, y_N \rangle| \leq C_N |Ay|^{1/2} \|y\|^{3/2}, \quad \text{for all } y \in D(A).
$$

This yields (see also the estimates below) $Ay_N, B_N(y_N) \in L^\infty(0, T; H)$.

Next we multiply Eq. (6.11) by $y_N$ and $Ay_N$, respectively. By (6.13) and (6.14) we get

$$
|y_N(t)|^2 + \nu \int_0^t \|y_N(s)\|^2 ds \leq |y_0|^2 + C \int_0^t |f(s)|^2 ds
$$

and (see (5.21)),

$$
\|y_N(t)\|^2 + \int_0^t |Ay_N(s)|^2 ds \leq \|y_0\|^2 + C \int_0^t \|y_N(s)\|^6 ds
$$

$$
+ C \int_0^t |f(s)|^2 ds, \quad \text{for all } t \in [0, T],
$$
respectively
\[
\|y_N(t)\|^2 + \int_0^t |A_N(s)|^2 \, ds
\]
\[
\leq \|y_0\|^2 + C \int_0^t \|y_N(s)\|^4 \, ds
\]
\[ + C \int_0^t |f(s)|^2 \, ds, \quad \text{for all } t \in [0, T], \tag{6.17}
\]
if \( n = 2 \). Here \( C \) is independent of \( N \).
This yields
\[
\|y_N(t)\| \leq C, \quad \text{for all } t \in [0, T] \tag{6.18}
\]
for \( n = 2 \) and
\[
\|y_N(t)\| \leq C, \quad \text{for } t \in [0, T_0] \subset [0, T] \tag{6.19}
\]
if \( n = 3 \).
Hence for \( N \geq C \) the solution \( y_N \) to (6.11) is a solution to
\[
\frac{d}{dt} y_N(t) + \nu A y_N + B_N y_N \ni f(t), \quad \text{a.e. } t \in [0, T], \tag{6.20}
\]
\[
y(0) = y_0
\]
and the results of Theorem 3.1 hold. This completes the proof.

6.2. Proof of Theorem 3.2
By (5.4) and the standard perturbation result it follows that the operator
\[
y \rightarrow \nu A y + B_N(y) + N_K^*(y) + \alpha_N y
\]
is maximal monotone and coercive in \( V \times V^* \) \( [2] \), and so its restriction to \( H \) is maximal monotone in \( H \times H \) (see [7, Chap. II, Example 2.3.7]) for \( \alpha_N \) large enough. This implies as above that the equation
\[
\frac{d}{dt} y(t) + \nu A y(t) + B_N(y(t)) + N_K^*(y(t)) \ni f(t), \quad \text{a.e. } t \in [0, T], \tag{6.22}
\]
with
\[
y(0) = y_0
\]
has a unique solution \( y_N \in W^{1,\infty}([0, T]; H) \cap L^2(0, T; V) \). As a matter of fact we have, as above,
\[
\frac{d^2 y_N(t)}{dt^2} + (\nu A_N y_N + B_N y_N(t))
\]
\[ + N_K^*(y_N(t)) - f(t) = 0, \quad \text{for all } t \in [0, T]. \tag{6.23}
\]
We have, since
\[
y(t)^2 + \int_0^t \|y(s)\|^2 ds \leq C, \quad \text{for all } t \in [0, T], \ N = 1, \ldots , \quad (6.24)
\]

Next we differentiate (6.22) formally and multiply the result by \(dy_N(t)/dt = y'_N(t)\) (the calculus can be made rigorous taking finite differences). We get
\[
\frac{1}{2} \frac{d}{dt} |y'_N(t)|^2 + \epsilon \|y'_N(t)\|^2 \leq \left( (B_N(y_N(t))', y'_N(t)) \\
+ \|f(t)\|y'_N(t)\), \quad \text{a.e. } t \in [0, T]. \quad (6.25)
\]

Here we used the fact that \((N_K(y(t)), y(t))\) is the limit of
\[
\frac{1}{h^2}(N_K(y(t+h)) - N_K(y(t)), y(t+h) - y(t)) \geq 0.
\]

We have, since \((B_N(y_N(t))', y'_N(t)) = b(y'_N(t), y_N(t), y_N(t)) + b(y_N(t), y'_N(t), y'_N(t)) \) and for \(n=2\)
\[
||((B_N(y_N(t))', y'_N(t)))|| = |b(y'_N(t), y'_N(t), y_N(t))| \\
\leq C\|y'_N(t)||y'_N(t)||y_N(t)|
\]
in \(t; \|y_N(t)\| \leq N\} \) and
\[
|((B_N(y_N(t))', y'_N(t))| \leq \frac{N^2}{\|y_N(t)\|^2} |b(y'_N(t), y'_N(t), y_N(t))| \\
+ \frac{2N^2\|y'_N(t)\|}{\|y_N(t)\|^3} |b(y_N(t), y'_N(t), y'_N(t))|
\]
in \(t; \|y_N(t)\| > N\}. \) This yields (for \(n=2\))
\[
|((B_N(y_N(t))', y'_N(t))| \leq C \frac{N^2}{\|y_N(t)\|^2} \|y_N(t)\|^2 \|y'_N(t)\|^2 \\
+ C \frac{2N^2\|y'_N(t)\|}{\|y_N(t)\|^3} \|y_N(t)\|^2 \|y'_N(t)\|^2.
\]

Thus
\[
|((B_N(y_N(t))', y'_N(t))| \leq C\|y_N(t)\| \|y_N(t)\| \|y'_N(t)\| \\
+ C\|y_N(t)\|^{1/2} \|y_N(t)\|^{1/2} \\
\|y_N(t)\|^{1/2} \|y'_N(t)\|^{3/2} \\
\leq \epsilon \|y_N(t)\|^2 + C(\|y_N(t)\|^2 \\
+ \|y_N(t)\|^2 \|y_N(t)\|^2 )\|y'_N(t)\|^2. \quad (6.26)
\]
Substitute the above results into (6.25), apply Gronwall’s inequality, and use the estimate (6.24) to get
\[ |y_N'(t)|^2 + \int_0^t \|y_N'(s)\|^2 ds \leq C, \quad \text{for all } N. \] (6.27)

Hence
\[ \|y_N(t)\| \leq C, \quad \text{for all } t \in [0, T] \] (6.28)
and so for N large enough, \( y(t) = y_N(t) \) is a solution to the Cauchy problem,
\[ \frac{d^+ y(t)}{dt} + (\nu Ay(t) + B_N(y(t)) + N_K^*(y(t)) - f(t)) = 0, \quad t \in [0, T], \] (6.29)
y(0) = y_0. \] (6.30)

This completes the proof.

6.3. Proof of Theorem 3.3. Consider the operator \( \mathcal{A}_N : D(\mathcal{A}_N) \to \mathbf{H} \) defined by
\[ \mathcal{A}_N y = \nu Ay + B_N(y) + \frac{1}{\lambda} P(my - P_K(my)), \quad D(\mathcal{A}_N) = D(A), \] (6.31)
where \( P : (L^2(\Omega))^n \to \mathbf{H} \) is the Helmholtz–Hodge projection. It is readily seen that the operator
\[ \mathbf{R} y = \frac{1}{\lambda} P(my - P_K(my)), \quad \text{for all } y \in \mathbf{H} \] (6.32)
is nonexpansive on \( \mathbf{H} \). Then by Lemma 5.1, \( \mathcal{A}_N + \alpha N I \) is m-accretive on \( \mathbf{H} \) and so the Cauchy problem
\[ \frac{dy}{dt} + \mathcal{A}_N(y) = f, \quad \text{a.e. } t \in [0, T], \] (6.33)
with
\[ y(0) = y_0, \] (6.34)
has a unique solution \( y_N^\lambda \in W^{1, \infty}([0, T]; \mathbf{H}) \cap L^\infty(0, T; D(A)) \). We have the obvious estimate
\[ |y_N^\lambda(t)|^2 + \nu \int_0^T \|y_N^\lambda(s)\|^2 ds \leq C, \quad \text{for all } \lambda > 0, \] (6.35)
where \( C \) is independent of \( \lambda \). Also for \( n = 2 \) we have
\[ \|y_N^\lambda(t)\|^2 \leq \|y_0\|^2 + C \int_0^t \|y_N^\lambda(s)\|^4 ds \]
\[ + C \int_0^t |f(s)|^2 ds + C \int_0^t |\mathbf{R} y_N^\lambda(s)|^2 ds, \quad \text{for all } t \in [0, T], \] (6.36)
\[ \leq C \left( \int_0^t \|y_N^\lambda(s)\|^4 ds + \frac{1}{\lambda^2} + 1 \right), \quad \text{for all } \lambda, N, t \in [0, T], \] (6.37)
respectively,
\[ \|y^N_\lambda(t)\|^2 \leq C \left( \int_0^t \|y^N_\lambda(s)\|^6 ds + \frac{1}{\lambda^2} + 1 \right), \]
for all \(\lambda, N, t \in [0, T]\) \hfill (6.38)
for \(n = 3\). Here \(C\) is independent of \(\lambda\) and \(N\). This yields
\[ \|y^N_\lambda(t)\| \leq C \left( 1 + \frac{1}{\lambda^2} \right), \quad \text{for all } t \in [0, T_0], \lambda > 0, \]
where \(T_0 = T\) if \(n = 2\) and \(0 < T_0 < T\) if \(n = 3\) and \(C\) is independent of \(\lambda\) and \(N\). Hence, for \(N \geq C(1 + \frac{1}{\lambda})\) we have
\[ B_N(y^N_\lambda(t)) = B(y^N_\lambda(t)), \quad \text{for all } t \in [0, T_0] \]
and so \(y_\lambda = y^N_\lambda\) is a solution to
\[ \frac{dy_\lambda}{dt} + \nu Ay_\lambda + B(y_\lambda) + R(y_\lambda) = f \quad \text{a.e. in } [0, T_0], \]
\[ y_\lambda(0) = y_0. \]
We have
\[ (Ry_\lambda, y_\lambda) \geq \frac{1}{2\lambda} d_K^2(my_\lambda), \quad \text{for all } t \in [0, T], \]
where \(d_K\) is the distance function to \(K\). Then by (6.4) we get
\[ |y_\lambda(t)|^2 + \frac{1}{\lambda} \int_0^T d_K^2(my_\lambda(t))dt \]
\[ + \int_0^T \|y_\lambda(s)\|^2 ds \leq C, \quad \text{for all } t \in [0, T], \]
where \(C\) is independent of \(\lambda\). This completes the proof.

7. CONCLUDING REMARKS

Remark 7.1. The particular type of V-ball cut-off type quantization we use here is guided closely by the global solvability theory of Navier–Stokes equations. In fact we may use an \(L^4\)-norm quantization of the form
\[ B_N(y) := \begin{cases} B(y) & \text{if } |y|_4 \leq N \\ \left( \frac{N}{|y|_4} \right)^4 B(y) & \text{if } |y|_4 > N \end{cases} \]
(7.1)
also to get m-accretivity (the authors thank J. L. Menaldi for pointing this out to them).
Remark 7.2. In this paper we considered two and three dimensional bounded domains (which will certainly cover periodic domains) and exterior domains. This type of m-accretive quantization also shows potential for other domains such as two and three dimensional manifolds [9] and also multi-channels and pipes (domains with many outlets at infinity) for which the boundaries are noncompact [20, 21].

REFERENCES