Exact Controllability for the Magnetohydrodynamic Equations

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Abstract

We study the local exact controllability of the steady state solutions of the magnetohydrodynamic equations. The main result of the paper asserts that the steady state solutions of these equations are locally controllable if they are smooth enough. We reduce the local exact controllability of the steady state solutions of the magnetohydrodynamic equations to the global exact controllability of the null solution of the linearized magnetohydrodynamic system via a fixed-point argument. The treatment of the reduced problem relies on two Carleman-type inequalities for the backward adjoint system. © 2003 Wiley Periodicals, Inc.

1 Introduction

This work is concerned with the local exact controllability of the magnetohydrodynamic (MHD) equations, which describe the motion of a viscous incompressible conducting fluid in a magnetic field and consist of a subtle coupling of the Navier-Stokes equations of viscous incompressible fluid flow and the Maxwell equations of electromagnetic field (see [4] or [17]).

Roughly speaking, the main result of this work, Theorem 2.1, amounts to saying that the steady state (stationary) solutions of the MHD equations are locally controllable provided that they are sufficiently smooth; that is, for such a steady state solution, if the initial data are sufficiently smooth and “close” to this solution, then there exist locally distributed (internal) controls such that the corresponding solutions of the MHD equations starting from these initial data reach the steady state solution in a finite time. For the Navier-Stokes and Boussinesq equations this result was previously established by O. Yu. Imanuvilov and A. V. Fursikov with O. Yu. Imanuvilov in [12] and [10], respectively. (For other literature
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concerning the controllability of the Navier-Stokes equations, we refer the reader to [1, 2, 7, 9, 11, 13].

The proof of Theorem 3.1 (which will be given in Section 4) is based on a fixed-point argument previously used in the controllability of the Navier-Stokes equations (see [1] or [2]). The main ingredients of the proof are two Carleman-type inequalities for the Stokes and dynamo equations (estimating their solutions in the entire domain by means of the restrictions of these solutions to the subdomain on which the controls are distributed). Much of the substance of this paper (Section 3) is devoted to the proof of the Carleman inequality for the dynamo equations, which also has an intrinsic interest.

The present result can be used to obtain the exact boundary controllability of the MHD equations. This will be presented in a subsequent paper.

Both internal distributed control and boundary control problems for magnetohydrodynamic systems arise in problems of plasma heating by radio and microwaves, plasma confinement by magnetic pinching, and active ionospheric heating by high-frequency radio waves (see [5]). The results of this paper can be considered as a first step towards a systematic control theoretic approach to these important technological challenges.

2 Main Result

Let \( \Omega \) be a bounded open set of \( \mathbb{R}^3 \) with a sufficiently smooth boundary \( \partial \Omega \), and let \( T > 0 \) be a fixed time. We suppose in addition that \( \Omega \) is simply connected and locally located on one side of \( \partial \Omega \). We set \( Q = \Omega \times (0, T) \). Consider an open subset \( \omega \) of \( \Omega \). The controlled MHD equations (with boundary and initial conditions) we deal with are the following:

\[
\frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y + \nabla p + \nabla \left( \frac{1}{2} B^2 \right) - (B \cdot \nabla) B = f + \chi_\omega u \quad \text{in} \quad Q,
\]

\[
\frac{\partial B}{\partial t} + \eta \nabla \text{curl}(\text{curl } B) + (y \cdot \nabla) B - (B \cdot \nabla) y = P(\chi_\omega v) \quad \text{in} \quad Q,
\]

\[
\nabla \cdot y = 0, \quad \nabla \cdot B = 0 \quad \text{in} \quad Q,
\]

\[
y = 0, \quad B \cdot N = 0, \quad (\text{curl } B) \times N = 0 \quad \text{on} \quad \Sigma = \partial \Omega \times (0, T),
\]

\[
y(\cdot, 0) = y_0, \quad B(\cdot, 0) = B_0 \quad \text{in} \quad \Omega.
\]

Here \( y = (y_1, y_2, y_3) : \Omega \times [0, T] \rightarrow \mathbb{R}^3 \) is the velocity vector field, \( p : \Omega \times [0, T] \rightarrow \mathbb{R} \) is the (scalar) pressure, and \( B = (B_1, B_2, B_3) : \Omega \times [0, T] \rightarrow \mathbb{R}^3 \) is the magnetic field. The vector functions \( u = (u_1, u_2, u_3) : \Omega \times [0, T] \rightarrow \mathbb{R}^3 \) and \( v = (v_1, v_2, v_3) : \Omega \times [0, T] \rightarrow \mathbb{R}^3 \) are controls, and \( \chi_\omega \) is the characteristic function of \( \omega \). We denote the variables of the functions \( y, p, B, u, \) and \( v \) by \( x = (x_1, x_2, x_3) \) (belonging to \( \Omega \)) and \( t \). The vector function \( f = (f_1, f_2, f_3) : \Omega \rightarrow \mathbb{R}^3 \)
is the known density of the external forces, and the vector fields \( y_0 : \Omega \to \mathbb{R}^3 \) and 
\( B_0 : \Omega \to \mathbb{R}^3 \) are the given initial velocity and magnetic fields, respectively. The 
operator \( P \) is the Leray projector. Applying \( P \) to the control \( \chi_0 v \) has the effect 
of “killing” its gradient part. Finally, \( v \) is the kinematic viscosity coefficient, \( \eta \) 
is the magnetic resistivity, and \( N \) is the exterior normal to the boundary. For the 
sake of simplicity (and without any loss of generality) we shall take \( v \) and \( \eta \) to 
be 1 in (2.1). The set \( \Omega \) may also be multiconnected, but in this case the values 
of a finite number of integrals of the magnetic field \( B \) should be prescribed. (We 
note here that in the first group of equations in system (2.1) the term 
\( (B \cdot \nabla)B - \nabla(\frac{1}{2} B^2) = (\text{curl} \, B) \times B \) represents the Lorentz force. The boundary conditions on 
\( B \) in (2.1) mathematically express the physical assumption that the boundary is 
perfectly conductive.) Let us notice that since \( \text{curl}(\text{curl} \, B) = -\Delta B + \text{grad}(\text{div} \, B) \), 
we can replace \( \text{curl}(\text{curl} \, B) \) in (2.1) by \(-\Delta B \).

In subsequent statements and considerations we need several function spaces. 
First of all, we shall use the Sobolev spaces \( H^m(\Omega) \) of functions that are in \( L^2(\Omega) \) 
together with their weak derivatives of order less than or equal to \( m \), where \( m \) are 
positive integers. Besides, we need the space \( H^{2,1}(Q) \) of functions in \( L^2(Q) \) whose 
first- and second-order weak derivatives with respect to the space variables \( x_1, x_2, \) 
and \( x_3 \) and first-order weak derivative with respect to \( t \) belong to \( L^2(Q) \), too. The 
space \( H^{1,1}(Q) \) is defined analogously. We shall also use some of the fractional-
order Sobolev spaces \( H^s(\Omega) \) and trace spaces \( H^s(\partial \Omega) \) (subspaces of \( L^2(\partial \Omega) \)) with 
\( s > 0 \). (For definitions and more information concerning these spaces, we refer the 
reader to [18].)

To set the MHD equations into a functional framework, we use the product 
space \((L^2(\Omega))^3\); we also need the product spaces \((L^2(Q))^3\), \((H^1(\Omega))^3\), \((H^2(\Omega))^3, \) 
\((H^{2,1}(\Omega))^3\), and \((H^{3/2}(\partial \Omega))^3\), endowed with the product norms. We denote by 
\((H^1_0(\Omega))^3\) the dual of the space \((H^1_0(\Omega))^3\) (of the vector fields in \((H^1(\Omega))^3\) that 
are equal to zero on the boundary). Finally, \( H^1(0, T; (L^2(\Omega))^3) \) is the space of the 
functions in \( L^2(0, T; (L^2(\Omega))^3) = (L^2(Q))^3 \) with the first-order weak derivative in 
\( L^2(0, T; (L^2(\Omega))^3) \) the space \( H^1(0, T; (H^1(\Omega))^3) \) is defined in a similar manner) 
and \( L^\infty(0, T; (H^2(\Omega))^3) \) is the space of all measurable and essentially bounded 
functions from \((0, T)\) to \((H^2(\Omega))^3\). The norms of all these spaces are denoted in 
the following way: \(| \cdot |_{L^2(\Omega)}^2, | \cdot |_{L^2(\Omega)}^3, | \cdot |_{H^2(\Omega)}^3, \) etc. Sometimes, for simplicity, 
when the domain \( \Omega \) is implied, we write these norms as \(| \cdot |_{L^2, \Omega}, | \cdot |_{H^2, \Omega} \), etc.

The MHD equations can be viewed as a system of two evolution equations 
in the space \( H \) of all weakly divergence-free vector fields in \((L^2(\Omega))^3\) that are 
tagential to the boundary in a weak sense, endowed with the \( L^2 \) norm:

\[
H = \{ y \in (L^2(\Omega))^3 : \text{div} \, y = 0 \text{ in } \Omega \text{ and } y \cdot N = 0 \text{ on } \partial \Omega \}.
\]
(In fact, \( H \) is the closure of the divergence-free vector fields in \( (C_c^\infty(\Omega))^3 \) in the \((L^2(\Omega))^3\) norm.) We also need the spaces
\[
V_1 = \{ y \in (H^1_0(\Omega))^3 : \text{div} \, y = 0 \text{ in } \Omega \},
V_2 = \{ B \in (H^1(\Omega))^3 : \text{div} \, B = 0 \text{ in } \Omega \text{ and } B \cdot N = 0 \text{ on } \partial \Omega \},
\]
both endowed with the \( H^1 \) norm. To write equations (2.1) as a system of evolution equations, we also need to define three operators. First, one knows that for each \( y \in (L^2(\Omega))^3 \) there exist unique \( y_1 \in H \) and \( y_2 = \nabla \phi \) for some \( \phi \in H^1(\Omega) \) such that \( y = y_1 + y_2 \). We define the orthogonal projection operator (Leray projector) \( P : (L^2(\Omega))^3 \to H \) by \( Py = y_1 \). The Stokes operator \( A_1 : D(A_1) \to H \) is defined as
\[
A_1 y = -P \Delta y \quad \text{for } y \in D(A_1),
\]
where \( \Delta \) is the Laplace operator and \( D(A_1) = (H^2(\Omega))^3 \cap V_1 \), endowed with the \( H^2 \) norm. Finally, we define the operator \( A_2 : D(A_2) \to H \) by
\[
A_2 B = \text{curl}(\text{curl} \, B) \quad \text{for } B \in D(A_2),
\]
where \( D(A_2) = \{ B \in (H^2(\Omega))^3 \cap V_2 : (\text{curl} \, B) \times N = 0 \text{ on } \partial \Omega \} \), endowed with the \( H^2 \) norm. (Mind that \( A_2 B \in H \) for \( B \in D(A_2) \).)

So, the MHD equations in (2.1) (along with the boundary conditions there) can be expressed as a system of two evolution equations in \( H \) in the following way:
\[
\begin{align*}
y' + A_1 y + P(y \cdot \nabla) y - P(B \cdot \nabla) B &= Pf + P(\chi \omega u), \\
B' + A_2 B + (y \cdot \nabla) B - (B \cdot \nabla) y &= P(\chi \omega v),
\end{align*}
\]
(2.2)

(Here and in what follows, we write \( y(t), B(t), u(t), \) and \( v(t) \) instead of \( y(\cdot, t), B(\cdot, t), u(\cdot, t), \) and \( v(\cdot, t) \) when we view them as elements of the space \((L^2(\Omega))^3\).) For more information concerning the mathematical setting and analysis of the MHD equations, we refer to [8, 14, 15, 16, 19].

Let \((y_0, B_0, p_0)\) be a steady state (equilibrium) solution of (2.1), i.e.,
\[
\begin{align*}
-\Delta y_e + (y_e \cdot \nabla) y_e + \nabla p_e + \nabla \left( \frac{1}{2} B_e^2 \right) - (B_e \cdot \nabla) B_e &= f \quad \text{in } \Omega, \\
\text{curl}(\text{curl} \, B_e) + (y_e \cdot \nabla) B_e - (B_e \cdot \nabla) y_e &= 0 \quad \text{in } \Omega,
\end{align*}
\]
(2.3)

\[
\begin{align*}
\nabla \cdot y_e &= 0, \quad \nabla \cdot B_e = 0 \quad \text{in } \Omega, \\
y_e &= 0, \quad B_e \cdot N = 0, \quad (\text{curl} \, B_e) \times N = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
(We refer the reader to [20] for the existence theory of these equations.) The main result of this paper, Theorem 2.1 below, asserts that if \( y_0 \) and \( B_0 \) are smooth enough, then they are locally controllable; that is, for \((y_0, B_0)\) sufficiently smooth
and “close” to \((y_e, B_e)\), there exist \(u, v, y, B, \) and \(p\) that satisfy equations (2.1) and 

\[ y(\cdot, T) = y_e, \ B(\cdot, T) = B_e. \]

**Theorem 2.1** Let \(\Omega\) be a simply connected, bounded, open subset of \(\mathbb{R}^3\) having a boundary \(\partial\Omega\) of class \(C^\infty\); suppose in addition that \(\Omega\) is locally located on one side of \(\partial\Omega\), and let \(\omega\) be an open subset of \(\Omega\). Let \(f \in (H^1(\Omega))^3\) and let \((y_e, B_e, p_e) \in ((H^3(\Omega))^3 \cap D(A_1)) \times ((H^3(\Omega))^3 \cap D(A_2)) \times H^1(\Omega)\) be a steady state solution of (2.3). Then there is \(\eta > 0\) such that for any \((y_0, B_0) \in D(A_1) \times D(A_2)\) satisfying

\[ |y_0 - y_e|_{(H^2(\Omega))^3} + |B_0 - B_e|_{(H^2(\Omega))^3} \leq \eta \]

there exists

\[(u, v, y, B, p) \in H^1(0, T; (L^2(\Omega))^3) \times H^1(0, T; (L^2(\Omega))^3) \times H^1(0, T; (L^2(\Omega))^3) \times H^1(0, T; (L^2(\Omega))^3) \times H^1(0, T; H^1(\Omega)) \]

that satisfies (2.1) or (2.2)) and

\[ y(x, T) = y_e(x), \ B(x, T) = B_e(x) \quad \text{a.e. } x \in \Omega. \]

It is easy to see that the problem of the local exact controllability of a stationary solution \((y_e, B_e)\) of system (2.1) is equivalent to the problem of the local exact controllability of the null solution of the system obtained by subtracting (2.1) and (2.3). Redenoting \(y - y_e, B - B_e,\) and \(p - p_e\) by \(y, B,\) and \(p,\) and the characteristic function \(\chi_\omega\) by \(m,\) the difference of (2.1) and (2.3) looks as follows:

\[
\frac{\partial y}{\partial t} - \Delta y + (y \cdot \nabla)y + (y_e \cdot \nabla)y + (y \cdot \nabla)y_e + \nabla p \\
+ \nabla \left( \frac{1}{2} B^2 \right) + \nabla (B_e \cdot B) - (B \cdot \nabla)B \\
- (B_e \cdot \nabla)B - (B \cdot \nabla)B_e = mu \quad \text{in } Q,
\]

\[ (4.4) \quad \frac{\partial B}{\partial t} + \text{curl(curl } B) + (y \cdot \nabla)B + (y_e \cdot \nabla)B \\
+ (y \cdot \nabla)B_e - (B \cdot \nabla)y - (B_e \cdot \nabla)y - (B \cdot \nabla)y_e = P(mv) \quad \text{in } Q,
\]

\[ \nabla \cdot y = 0, \ \nabla \cdot B = 0 \quad \text{in } Q,
\]

\[ y = 0, \ B \cdot N = 0, \ (\text{curl } B) \times N = 0 \quad \text{on } \Sigma,
\]

\[ y(\cdot, 0) = y^0 = y_0 - y_e, \ B(\cdot, 0) = B^0 = B_0 - B_e \quad \text{in } \Omega.
\]

By Kakutani’s fixed-point theorem, the local exact controllability of the null solution of system (2.4) will be reduced to the global exact controllability of the
null solution of the following linearized system:

\[
\begin{align*}
\frac{\partial y}{\partial t} & - \Delta y + ((w + y_e) \cdot \nabla) y + (y \cdot \nabla) y_e + E \cdot (\nabla B) \\
+ \nabla (B_e \cdot B) - ((E + B_e) \cdot \nabla) B - (B \cdot \nabla) B_e + \nabla p = mu & \quad \text{in } Q, \\
\frac{\partial B}{\partial t} + \text{curl(curl } B) + P((w + y_e) \cdot \nabla) B + (y \cdot \nabla) B_e \\
- ((E + B_e) \cdot \nabla) y - (B \cdot \nabla) y_e = P(mv) & \quad \text{in } Q, \\
\nabla \cdot y = 0, \quad \nabla \cdot B = 0 & \quad \text{in } Q, \\
y = 0, \quad B \cdot N = 0, \ (\text{curl } B) \times N = 0 & \quad \text{on } \Sigma, \\
y(\cdot, 0) = y^0, \ B(\cdot, 0) = B^0 & \quad \text{in } \Omega, 
\end{align*}
\]

(2.5)

where

\[
(E \cdot (\nabla B))_i = \sum_{j=1}^{3} E_j \frac{\partial B_j}{\partial x_i},
\]

and \((w, E)\) is a fixed pair having the expected regularity of \((y, B)\) in (2.1) (asserted in Theorem 2.1). On the other hand, the null controllability of system (2.5) is equivalent to the observability of its backward dual system, which will be proven in Section 4 by using Carleman-type inequalities for the backward Stokes and dynamo equations.

### 3 Carleman Inequalities for the Stokes and Dynamo Equations

The crucial point of our arguments is the use of the Carleman inequalities for the backward Stokes equations with the no-slip boundary condition:

\[
\begin{align*}
\frac{\partial z}{\partial t} + \Delta z + \nabla q = g & \quad \text{in } Q = \Omega \times (0, T), \\
\nabla \cdot z = 0 & \quad \text{in } Q, \\
z = 0 & \quad \text{on } \Sigma = \partial \Omega \times (0, T),
\end{align*}
\]

(3.1)

and for the following backward dynamo-type equations in domains with perfectly conductive boundary:

\[
\begin{align*}
\frac{\partial C}{\partial t} + \Delta C = PG & \quad \text{in } Q, \\
\nabla \cdot C = 0 & \quad \text{in } Q, \\
C \cdot N = 0, \ (\text{curl } C) \times N = 0 & \quad \text{on } \Sigma.
\end{align*}
\]

(3.2)

Essentially, these inequalities are a priori estimates for solutions of (3.1) and (3.2) by means of the right-hand side terms and the restrictions of solutions on \(Q_{\omega} = \omega \times (0, T)\).

Let \(\Omega\) and \(\omega\) be open sets as in the statement of Theorem 2.1, and let \(\omega_0\) and \(\omega_1\) be arbitrary open subsets of \(\omega\) such that \(\omega_0 \subseteq \omega_1 \subseteq \omega\). To express the Carleman
inequalities, we need two auxiliary functions defined by means of a function $\psi \in \mathcal{C}^2(\bar{\Omega})$ that satisfies the following conditions:

\begin{equation}
\psi > 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial \Omega, \quad |\nabla \psi| > 0 \text{ in } \Omega \setminus \omega_0.
\end{equation}

To be able to make a suitable estimate of the pressure in (3.1), we need a function $\psi$ as above which also has a certain form in a neighborhood of the boundary.

Let $U$ be a neighborhood of $\partial \Omega$, and set $D = \Omega \cap U$. Suppose that $\partial D = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 = \partial \Omega$ and $\Gamma_1$ is diffeomorphic with $\Gamma_0$. It is easy to see that there exists a function $\theta \in \mathcal{C}^\infty(\bar{D})$ such that

\begin{equation}
0 \leq \theta \leq 1, \quad |\nabla \theta| > 0 \text{ in } \bar{D}, \quad \theta = 0 \text{ on } \Gamma_1, \quad \theta = 1 \text{ on } \Gamma_0.
\end{equation}

Moreover, it is known (see [2, 12]) that it is possible to choose such a function $\theta$ such that, for the surface $\Gamma_t = \{x \in D : \theta(x) = 1 - t\}$, $t \in [0, T]$, the following interpolation inequality holds:

\begin{equation}
\int_{\Gamma_t} |v|^2 \, d\sigma \leq c |v|_{L^2(\Gamma_0)}^2 |v|_{L^2(\Gamma_1)}^2
\end{equation}

for all $v \in H^1(D)$ that satisfy

\[ \Delta v = 0 \text{ in } D, \quad \frac{\partial v}{\partial N} = 0 \text{ on } \Gamma_1, \]

and $t \in [0, T]$. (Here the constant $c > 0$ does not depend on $t$.)

One can show (see [2, 12]) that there exists a function $\psi \in \mathcal{C}^\infty(\bar{\Omega})$ satisfying (3.3) such that $\psi = 1 - \theta$ in $D$, where $D$ is an open set as above and $\theta$ is a function in $\mathcal{C}^\infty(\bar{D})$ that satisfies (3.4) and (3.5). Let us fix (in what follows) such a function $\psi$.

Now, we set

\[ \varphi(x, t) = \frac{e^{\lambda \psi(x)}}{(t(T-t))^2}, \quad \alpha(x, t) = \frac{e^{\lambda \psi(x)} - e^{2\lambda |\psi|_{\mathcal{C}(\bar{\Omega})}}}{(t(T-t))^2}, \]

where $\lambda > 0$. We denote by $\varphi^*(t)$ and $\alpha^*(t)$ the values taken by $\varphi$ and $\alpha$ on the boundary $\partial \Omega$ (where $\psi = 0$):

\[ \varphi^*(t) = \frac{1}{(t(T-t))^2}, \quad \alpha^*(t) = \frac{1 - e^{2\lambda |\psi|_{\mathcal{C}(\bar{\Omega})}}}{(t(T-t))^2}. \]

We shall also use the functions $\overline{\varphi}$ and $\overline{\alpha}$ defined as

\[ \overline{\varphi}(x, t) = \frac{e^{-\lambda \psi(x)}}{(t(T-t))^2}, \quad \overline{\alpha}(x, t) = \frac{e^{-\lambda \psi(x)} - e^{2\lambda |\psi|_{\mathcal{C}(\bar{\Omega})}}}{(t(T-t))^2}. \]

We set $Q_{\omega_0} = \omega_0 \times (0, T)$ and $Q_{\omega_1} = \omega_1 \times (0, T)$, and we recall that $Q_\omega = \omega \times (0, T)$. Now we can present the two Carleman inequalities.
THEOREM 3.1 Let $\Omega$ be a simply connected, bounded, open subset of $\mathbb{R}^3$ having the boundary $\partial \Omega$ of class $C^\infty$; suppose in addition that $\Omega$ is locally located on one side of $\partial \Omega$, and let $\omega$ be an open subset of $\Omega$. Then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ one can find $s_0(\lambda) > 0$ and $c(\lambda) > 0$ that for $s > s_0(\lambda)$ the following inequality holds:

$$
\int_Q e^{2s\varphi} \left( \frac{1}{s \varphi} \left( \left| \frac{\partial z}{\partial t} \right|^2 + \sum_{i,j=1}^3 \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \right) + s \varphi |\nabla z|^2 + s^3 \varphi^3 |z|^2 \right) dx dt
$$

$$
\leq c(\lambda) \left( \int_Q e^{2s\varphi} (|g|^2 + |\text{div } g|^2) dx dt + \int_Q e^{2s\varphi} s \varphi^* |g|^2 dx dt + \int_0^T e^{-\delta s\varphi} |g(\cdot, t)|^2_{(H^1_0(\omega))} dt + \int_{Q_\omega} e^{2s\varphi} s^3 \varphi^3 |z|^2 dx dt \right)
$$

(3.6)

for all $g \in L^2(0, T; (H^1(\Omega))^3)$ and all corresponding solutions $(z, p) \in (H^{2,1}(Q))^3 \times L^2(0, T; H^1(\Omega))$ of system (3.1), and for some $\delta > 0$.

Essentially, inequality (3.6) was established by O. Yu. Imanuvilov in [12]. The present form is taken from [1] (see also [2]).

THEOREM 3.2 If $\Omega$ and $\omega$ are as in the statement of Theorem 3.1, then there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ one can find $s_0(\lambda) > 0$ and $c(\lambda) > 0$ that for $s > s_0(\lambda)$ the following inequality holds:

$$
\int_Q e^{2s\varphi} \left( \frac{1}{s \varphi} \left( \left| \frac{\partial C}{\partial t} \right|^2 + \sum_{i,j=1}^3 \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) + s \varphi |\nabla C|^2 + s^3 \varphi^3 |C|^2 \right) dx dt
$$

$$
\leq c(\lambda) \left( \int_Q e^{2s\varphi} (|G|^2 + |\text{div } G|^2) dx dt + \int_0^T e^{-\delta s\varphi} |G(\cdot, t)|^2_{(H^1_0(\omega))^3} dt + \int_{Q_\omega} e^{2s\varphi} s^3 \varphi^3 |C|^2 dx dt \right)
$$

(3.7)

for all first-order linear differential operators $G : (H^{1,1}(Q))^3 \to (L^2(Q))^3$ satisfying (for some $c > 0$)

$$
|G(C) \cdot N| \leq c|C| \text{ on } \Sigma \text{ for } C \in (H^{1,1}(Q))^3,
$$

all corresponding solutions $C \in (H^{2,1}(Q))^3$ of system (3.2), and for some $\delta > 0$. 
(For simplicity, in (3.7) and in the following we write $G$ instead of $G(C)$.)

The proof of Theorem 3.2 follows the same lines as the proof of Theorem 3.1. However, the different boundary conditions require a careful treatment of the surface integrals arising in the integrations by parts. Since, in a first stage of the proof, we are concerned with obtaining a version of inequality (3.7) containing only the solution and its gradient in the left-hand side, it is desirable to remove all the surface integrals that contain first-order derivatives. Most of them can be eliminated by a trick used in deriving the Carleman inequality for linear parabolic equations with Neumann-type boundary conditions (note that the second boundary condition in (3.2) is of the first order). The main effort will be concentrated on expressing the integrals which remain unmoved by surface integrals that do not contain any derivative of the solution. This will be done by fully using all our specific boundary conditions in a suitable manner. For this reason we shall develop the proof in all its details.

PROOF OF THEOREM 3.2: The proof contains two parts. First we shall establish (as we have already said) a variant of inequality (3.7) that does not contain the time derivative and the second-order derivatives of the solution. Then, in the second part, we shall estimate the $L^2$ norms of these derivatives by means of the $L^2$ norm of the gradient of the solution. These taken together will give inequality (3.7). Of course, it suffices to prove (3.7) only for smooth solutions (because the solutions in $(H^{2,1}(Q))^3$ of (3.2) can be approximated in the $(H^{2,1}(Q))^3$ norm by smooth solutions).

To obtain a Carleman inequality containing only the $L^2$ norms of solution and its gradient in the left-hand side, we change the unknown of system (3.2) in a convenient manner: Set $D = e^{s\alpha} C$. Changing $C$ by $D$, system (3.2) becomes

$$
\begin{align*}
\frac{\partial D}{\partial t} + \Delta D + \lambda^2 \varphi |\nabla \psi|^2 D - 2s\lambda \varphi (\nabla \psi \cdot \nabla) D
&= -s\lambda^2 \varphi |\nabla \psi|^2 D - s\lambda \varphi \Delta \psi D - s \frac{\partial \alpha}{\partial t} D = e^{s\alpha} PG & \text{in } Q, \\
\Div D &= s\lambda \varphi (\nabla \psi \cdot D) & \text{in } Q, \\
D \cdot N &= 0, \quad \text{(curl } D) \times N = -s\lambda \varphi |\nabla \psi| D & \text{on } \Sigma, \\
D(\cdot, 0) &= D(\cdot, T) = 0 & \text{in } \Omega.
\end{align*}
$$

(3.8)

We divide the terms in the left-hand side of equations (3.8) except $\partial D/\partial t$ into two blocks described with the aid of the differential operators $P(t)$ and $R(t)$:

$$
P(t)D = -\Delta D - s^{2}\lambda^{2} \varphi |\nabla \psi|^2 D - s\lambda^2 \varphi |\nabla \psi|^2 D + s\lambda \varphi \Delta \psi D + s \frac{\partial \alpha}{\partial t} D, \\
R(t)D &= -2s\lambda \varphi (\nabla \psi \cdot \nabla) D - 2s\lambda^2 \varphi |\nabla \psi|^2 D.
$$
So we can write equations (3.8) as

\begin{equation}
\frac{\partial D}{\partial t} + R(t)D - P(t)D = e^{\alpha t} \, PG \quad \text{in } Q.
\end{equation}

Multiplying equation (3.9) by itself and integrating over $Q$, we obtain

\begin{equation}
\int_Q \left( \left| \frac{\partial D}{\partial t} + R(t)D \right|^2 + |P(t)D|^2 \right) \, dx \, dt - 2 \int_Q P(t)D \cdot R(t)D \, dx \, dt = \nonumber \end{equation}

\begin{equation}
2 \int_Q \frac{\partial D}{\partial t} \cdot P(t)D \, dx \, dt + \int_Q e^{2\alpha t} |PG|^2 \, dx \, dt.
\end{equation}

We set

\begin{equation}
I = -\int_Q P(t)D \cdot R(t)D \, dx \, dt, \quad J = \int_Q \frac{\partial D}{\partial t} \cdot P(t)D \, dx \, dt.
\end{equation}

From (3.10) we have

\begin{equation}
2I \leq 2J + \int_Q e^{2\alpha t} |PG|^2 \, dx \, dt.
\end{equation}

The desired Carleman inequality will follow from inequality (3.11). To this end, we have to eliminate the second-order derivatives of $D$ from $I$ and $J$ by integration by parts. Doing suitable estimates we shall keep only integrals containing $|D|^2$ and $|\nabla D|^2$ on both sides of (3.11). The choice of the blocks $P(t)D$ and $R(t)D$ is guided only by our desire to obtain the powers of $s$, $\lambda$, and $\varphi$ in the left-hand side of the final inequality (deriving from (3.11)) greater than the analogous powers corresponding to similar terms in the right-hand side. This fact is crucial in transforming integration over $Q$ to integration over $Q_{\omega_0}$ in the right-hand side of the final inequality.

First we shall deal with $I$. It can be written as a sum of ten terms (in fact, integrals): $I = \sum_{i=1}^{10} I_i$. Let us see them one by one.

We notice that

\begin{equation}
N = -\frac{\nabla \psi}{|\nabla \psi|},
\end{equation}
because $\psi = 0$ on $\partial \Omega$. Two integrations by parts (the first of them being a use of Green’s formula) together with (3.12) give

$$I_1 = -2s\lambda \int_Q \varphi (\nabla \psi \cdot \nabla) D \cdot \Delta D \, dx \, dt$$

$$= 2s\lambda^2 \sum_{i=1}^{3} \int_Q \varphi (\nabla \psi \cdot \nabla D_i)^2 \, dx \, dt + 2s\lambda \sum_{i=1}^{3} \int_Q \varphi \sum_{j,k=1}^{3} \frac{\partial^2 \psi}{\partial x_j \partial x_k} \frac{\partial D_i}{\partial x_j} \frac{\partial D_i}{\partial x_k} \, dx \, dt$$

$$- s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla D|^2 \, dx \, dt - s\lambda \int_Q \varphi \Delta \psi |\nabla D|^2 \, dx \, dt$$

$$+ 2s\lambda \int_{\Sigma} \varphi |\nabla \psi| \sum_{i=1}^{3} \left( \frac{\partial D_i}{\partial N} \right)^2 \, d\sigma \, dt - s\lambda \int_{\Sigma} \varphi |\nabla \psi| |\nabla D|^2 \, d\sigma \, dt.$$ 

It is easy to see that

$$I_1 \geq -s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla D|^2 \, dx \, dt$$

$$- 9 \max_{1 \leq j,k \leq 3} \sup_{\Omega} \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right| s\lambda \int_Q \varphi |\nabla D|^2 \, dx \, dt$$

$$+ 2s\lambda \int_{\Sigma} \varphi |\nabla \psi| \sum_{i=1}^{3} \left( \frac{\partial D_i}{\partial N} \right)^2 \, d\sigma \, dt - s\lambda \int_{\Sigma} \varphi |\nabla \psi| |\nabla D|^2 \, d\sigma \, dt.$$ 

Using Green’s formula once again, we obtain

$$I_2 = -2s\lambda^2 \int_Q \varphi |\nabla \psi|^2 D \cdot \Delta D \, dx \, dt$$

$$= 2s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla D|^2 \, dx \, dt$$

$$+ 2s\lambda^3 \sum_{i=1}^{3} \int_Q \varphi |\nabla \psi|^2 D_i \nabla \psi \cdot \nabla D_i \, dx \, dt$$

$$+ 4s\lambda^2 \sum_{i=1}^{3} \int_Q \varphi D_i \sum_{j,k=1}^{3} \frac{\partial^2 \psi}{\partial x_j \partial x_k} \frac{\partial \psi}{\partial x_j} \frac{\partial D_i}{\partial x_k} \, dx \, dt$$

$$- 2s\lambda^2 \int_{\Sigma} \varphi |\nabla \psi|^2 \sum_{i,j=1}^{3} D_i \frac{\partial D_j}{\partial x_j} N_j \, d\sigma \, dt.$$
It is easy to check that
\begin{equation}
2s\lambda^3 \int_Q \varphi |\nabla \psi|^2 D_i \nabla \psi \cdot \nabla D_i \, dx \, dt \geq \\
- 4 \sup_{\Omega} |\nabla \psi|^4 s\lambda^4 \int_Q \varphi |D|^2 \, dx \, dt - \frac{1}{4} s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla D|^2 \, dx \, dt,
\end{equation}
(3.15)

\begin{equation}
4s\lambda^2 \sum_{i=1}^3 \int_Q \varphi D_i \sum_{j,k=1}^3 \frac{\partial^2 \psi}{\partial x_j \partial x_k} \frac{\partial \psi}{\partial x_j} \frac{\partial D_i}{\partial x_k} \, dx \, dt \geq \\
- 16 \sup_{\Omega} \sum_{j,k=1}^3 \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right|^2 s\lambda^2 \int_Q \varphi |D|^2 \, dx \, dt - \frac{1}{4} s\lambda^2 \int_Q \varphi |\nabla \psi|^2 |\nabla D|^2 \, dx \, dt.
\end{equation}
(3.16)

Now let us treat the surface integral in (3.14). As we have already said after the statement of Theorem 3.2, we have to eliminate all the surface integrals containing first-order derivatives of $D$ because, via suitable trace theorems, these produce integrals over $Q$ containing undesirable second-order derivatives of $D$. To anticipate our subsequent considerations, let us say that all the surface integrals that are multiplied by powers of $\lambda$ having odd exponents can be cancelled by a trick used in establishing the Carleman inequality for linear parabolic equations with Neumann boundary conditions. It remains to treat the surface integrals that are multiplied by powers with even exponents.

It is remarkable that the surface integral in (3.14) can be expressed by surface integrals that do not contain any derivative of $D$. Indeed, taking the second boundary condition in (3.8) and (3.12) into account, we have
\begin{equation}
- \sum_{i,j=1}^3 D_i \frac{\partial D_i}{\partial x_j} N_j = -((\text{curl } D) \times N) \cdot D - \sum_{i,j=1}^3 D_i \frac{\partial D_j}{\partial x_i} N_j \\
= s\lambda \varphi |\nabla \psi| \nabla D^2 + \sum_{i,j=1}^3 |\nabla \psi|^{-1} D_i \frac{\partial D_j}{\partial x_i} \frac{\partial \psi}{\partial x_j} \\
= s\lambda \varphi |\nabla \psi| \nabla D^2 + |\nabla \psi|^{-1} D \cdot \nabla (D \cdot \nabla \psi) \\
- |\nabla \psi|^{-1} \sum_{i,j=1}^3 D_i D_j \frac{\partial^2 \psi}{\partial x_i \partial x_j}.
\end{equation}
(3.17)

Since, by the first boundary condition in (3.8), $D \cdot \nabla \psi = -|\nabla \psi|(D \cdot N) = 0$ on $\partial \Omega$, the vectors $\nabla (D \cdot \nabla \psi)$ and $N$ have the same direction at the same point of $\partial \Omega$. Thus, again using the fact that $D \cdot N = 0$ on $\partial \Omega$, we have in (3.17)
\[ D \cdot \nabla (D \cdot \nabla \psi) = 0 \quad \text{on } \partial \Omega, \]
and consequently,
\begin{equation}
-2s\lambda^2 \int_\Sigma \varphi |\nabla \psi|^2 \sum_{i,j=1}^3 D_i \frac{\partial D_j}{\partial x_j} N_j \, d\sigma \, dt =
\end{equation}
\begin{equation}
2s^2\lambda^3 \int_\Sigma \varphi^2 |\nabla \psi|^3 |D|^2 \, d\sigma \, dt - 2s\lambda^2 \int_\Sigma \varphi |\nabla \psi| \sum_{i,j=1}^3 \frac{\partial^2 \psi}{\partial x_i \partial x_j} D_i D_j \, d\sigma \, dt .
\end{equation}

The first integral in the right-hand side of (3.18) is multiplied by a power of \( \lambda \) with odd exponent, so, as we have said, it will disappear by a suitable treatment that has no effect on the second integral multiplied by \( \lambda^2 \). By using a trace theorem, the second surface integral can be replaced by two integrals over \( Q \) containing \(|D|^2\) and \(|\nabla D|^2\), respectively, in the right-hand side of the final inequality. Certainly, these integrals appear multiplied by \( s\lambda^2 \). But we shall see further that the greatest exponents of the powers of \( s \) and \( \lambda \) that accompany the integrals over \( Q \) containing \(|\nabla D|^2\) in the left-hand side are exactly 1 and 2, respectively, as before and not greater than we need. To avoid this situation, we shall use the trace theorem in a suitable interpolation space and then shall apply the corresponding interpolation inequality.

Clearly, we have
\begin{equation}
-2s\lambda^2 \int_\Sigma \varphi |\nabla \psi| \sum_{i,j=1}^3 \frac{\partial^2 \psi}{\partial x_i \partial x_j} D_i D_j \, d\sigma \, dt \geq
\end{equation}
\begin{equation}
2s^2\lambda^2 \max_{1 \leq i \leq 3} \sup_{aΩ} \left( |\nabla \psi| \sum_{j=1}^3 \left| \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right| \right) \sum_{i=1}^3 \int_0^T \varphi^\ast(t) \left( \int_{\partial Ω} D_i^2 \, d\sigma \right) \, dt .
\end{equation}

Let \( \frac{1}{2} \leq \alpha < 1 \) and \( 0 < \beta < \alpha \). Using the trace theorem, an interpolation inequality, and Young’s inequality, we obtain
\begin{equation}
s\lambda^2 \int_{\partial \Omega} D_i^2 \, d\sigma \leq c_1 s\lambda^2 |D_i|_{H^{\alpha-1/2}(\partial \Omega)}^2 \leq c_2 s\lambda^2 |D_i|_{H^\alpha(\Omega)}^2 \leq c_3 s\lambda^2 |D_i|_{L^2(\Omega)}^{2(1-\alpha)} |D_i|_{H^1(\Omega)}^{2\alpha}
\end{equation}
\begin{equation}
= c_3 \left( (s\lambda^2)^{1-\beta/\alpha} |D_i|_{L^2(\Omega)}^2 \right)^{1-\alpha} \left( (s\lambda^2)^{\beta/\alpha} |D_i|_{H^1(\Omega)}^2 \right)^{\alpha} \leq c_3 (1-\alpha) (s\lambda^2)^{1-\beta/\alpha} |D_i|_{L^2(\Omega)}^2 + c_3 \alpha (s\lambda^2)^{\beta/\alpha} |D_i|_{L^2(\Omega)}^2 + c_3 \alpha (s\lambda^2)^{\beta/\alpha} |\nabla D_i|_{L^2(\Omega)}^2 .
\end{equation}
where $c_1$, $c_2$, and $c_3$ are positive constants that depend only on $\Omega$. Taking $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$, inequalities (3.19) and (3.20) give

$$
-2s\lambda^2 \int_\Sigma \varphi |\nabla \psi| \sum_{i,j=1}^3 \frac{\partial^2 \psi}{\partial x_i \partial x_j} D_i D_j d\sigma dt
$$

$$
\geq -2c_3 \max_{1 \leq i \leq 3} \sup_{\partial \Omega} \left( |\nabla \psi| \sum_{j=1}^3 \left| \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right| \right)
$$

$$
\times \left( s^{3/2} \lambda^3 \int_Q \varphi |D|^2 dx dt + s\lambda \int_Q \varphi |\nabla D|^2 dx dt \right),
$$

where $c_3$ is the constant in (3.20).

Integrating by parts, we obtain after some calculation that

$$
I_3 = -2s^3 \lambda^3 \int_Q \varphi^3 |\nabla \psi|^2 D \cdot (\nabla \psi \cdot \nabla) D dx dt
$$

$$
= 3s^3 \lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |D|^2 dx dt
$$

$$
+ 2s^3 \lambda^3 \int_Q \varphi^3 \sum_{j,k=1}^3 \frac{\partial^2 \psi}{\partial x_j \partial x_k} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_k} |D|^2 dx dt
$$

$$
+ s^3 \lambda^3 \int_Q \varphi^3 |\nabla \psi|^2 \Delta \psi |D|^2 dx dt
$$

$$
- s^3 \lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 |D|^2 d\sigma dt
$$

$$
\geq 3s^3 \lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |D|^2 dx dt
$$

$$
- 27 \max_{1 \leq j,k \leq 3} \sup_{\Omega} \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right| \left( \max_{1 \leq j \leq 3} \sup_{\Omega} \left| \frac{\partial \psi}{\partial x_j} \right| \right)^2 s^3 \lambda^3 \int_Q \varphi^3 |D|^2 dx dt
$$

$$
- s^3 \lambda^3 \int_\Sigma \varphi^3 |\nabla \psi|^2 \frac{\partial \psi}{\partial N} |D|^2 d\sigma dt.
$$

Let us point out that here we have the integral over $Q$ containing $|D|^2$, which is multiplied by the dominated powers of $s$, $\lambda$, and $\varphi$, that is, $s^3$, $\lambda^4$, and $\varphi^3$. The coefficient $3$ of $s^3 \lambda^4 \varphi^3$ will still decrease if we add $I_3$ to

$$
I_4 = -2s^3 \lambda^4 \int_Q \varphi^3 |\nabla \psi|^4 |D|^2 dx dt.
$$
We have as before
\[ I_5 = -2s^2\lambda^3 \int_Q \varphi^2 \left| \nabla \psi \right|^2 D \cdot (\nabla \psi \cdot \nabla) D \, dx \, dt \]
\[ \geq 2s^2\lambda^4 \int_Q \varphi^2 \left| \nabla \psi \right|^4 |D|^2 \, dx \, dt \]
(3.24)
\[-27 \max_{1 \leq j, k \leq 3} \sup_{\Omega} \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right| \left( \max_{1 \leq j \leq 3} \sup_{\Omega} \left| \frac{\partial \psi}{\partial x_j} \right| \right)^2 s^2\lambda^3 \int_Q \varphi^2 |D|^2 \, dx \, dt \]
\[-s^2\lambda^3 \int_{\Sigma} \varphi^2 \left| \nabla \psi \right|^2 \frac{\partial \psi}{\partial N} |D|^2 \, d\sigma \, dt .\]

We also set
(3.25) \[ I_6 = -2s^2\lambda^4 \int_Q \varphi^2 \left| \nabla \psi \right|^4 |D|^2 \, dx \, dt .\]

Further, it is easy to see that
\[ I_7 = 2s^2\lambda^2 \int_Q \varphi^2 \Delta \psi D \cdot (\nabla \psi \cdot \nabla) D \, dx \, dt \]
\[ \geq -3 \sup_{\Omega} (\Delta \psi)^2 s^3\lambda^3 \int_Q \varphi^3 |D|^2 \, dx \, dt \]
(3.26)
\[-\max_{1 \leq i \leq 3} \sup_{\Omega} \left( \frac{\partial \psi}{\partial x_i} \right)^2 s\lambda \int_Q \varphi |\nabla D|^2 \, dx \, dt \]
and
\[ I_8 = 2s^2\lambda^3 \int_Q \varphi^2 \left| \nabla \psi \right|^2 \Delta \psi |D|^2 \, dx \, dt \]
\[ \geq -2 \sup_{\Omega} (|\nabla \psi|^2 |\Delta \psi|) s^2\lambda^3 \int_Q \varphi^2 |D|^2 \, dx \, dt .\]
(3.27)

Finally, after integration by parts and some calculation, we obtain
\[ I_9 = 2s^2\lambda \int_Q \varphi \frac{\partial \alpha}{\partial t} D \cdot (\nabla \psi \cdot \nabla) D \, dx \, dt \]
\[ = -2s^2\lambda^2 \int_Q \varphi |\nabla \psi|^2 \left| \frac{\partial \alpha}{\partial t} \right| |D|^2 \, dx \, dt \]
\[ + 2s^2\lambda^2 \gamma(\lambda) \int_Q \varphi |\nabla \psi|^2 \frac{1}{(t(T-t))^3} (T - 2t)|D|^2 \, dx \, dt \]
\[-s^2\lambda \int_Q \varphi \frac{\partial \alpha}{\partial t} \Delta \psi |D|^2 \, dx \, dt + s^2\lambda \int_{\Sigma} \varphi \frac{\partial \alpha}{\partial t} \frac{\partial \psi}{\partial N} |D|^2 \, d\sigma \, dt .\]
where \( \gamma(\lambda) = e^{2\lambda|\psi|e^{\lambda}} \). Adding

\[
I_{10} = 2s^2\lambda^2 \int_{\Omega} \varphi |\nabla \psi|^2 \frac{\partial \alpha}{\partial t} |D|^2 \, dx \, dt ,
\]

we have

\[
I_9 + I_{10} \geq -2s^2\lambda^2 \gamma(\lambda)T \sup_{\Omega} |\nabla \psi|^2 \int_{\Omega} \varphi^2 |D|^2 \, dx \, dt \\
- 2s^2\lambda T \sup_{\Omega} |\Delta \psi| \int_{\Omega} \varphi^2 |D|^2 \, dx \, dt \\
- 2s^2\lambda \gamma(\lambda)T \sup_{\Omega} |\Delta \psi| \int_{\Omega} \varphi^2 |D|^2 \, dx \, dt \\
+ s^2\lambda \int_{\Sigma} \varphi \frac{\partial \alpha}{\partial t} \frac{\partial \psi}{\partial N} |D|^2 \, d\sigma \, dt .
\]

(3.28)

Now let us estimate the integral \( J \). First we eliminate the second-order derivatives of \( D \) in \( J \) by an adequate integration by parts. Removing also the derivative of \( D \) with respect to \( t \) and taking the boundary conditions in (3.8) into account, we have

\[
- \int_{\Omega} \Delta D \cdot \frac{\partial D}{\partial t} \, dx \, dt \\
= \int_{\Omega} \text{curl(curl} D) \cdot \frac{\partial D}{\partial t} \, dx \, dt - \int_{\Omega} \text{grad(div} D) \cdot \frac{\partial D}{\partial t} \, dx \, dt \\
= \int_{\Omega} \text{curl} D \cdot \text{curl}\left( \frac{\partial D}{\partial t} \right) \, dx \, dt + \int_{\Sigma} \left( \text{curl} D \times \frac{\partial D}{\partial t} \right) \cdot N \, d\sigma \, dt \\
\]

(3.29)

\[
+ \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} |\text{div} D|^2 \, dx \, dt - \int_{\Sigma} \text{div} D \frac{\partial}{\partial t} (D \cdot N) d\sigma \, dt \\
= \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} |\text{curl} D|^2 \, dx \, dt + \frac{1}{2} s\lambda \int_{\Sigma} \varphi |\nabla \psi| \frac{\partial}{\partial t} |D|^2 \, d\sigma \, dt \\
= - \frac{1}{2} s\lambda \int_{\Sigma} \frac{\partial \varphi}{\partial t} |\nabla \psi| |D|^2 \, d\sigma \, dt .
\]
Another integration by parts with respect to \( t \) together with (3.29) yields
\[
\begin{align*}
J &= -\frac{1}{2} s \lambda \int_{\Sigma} \frac{\partial \phi}{\partial t} |\nabla \psi| |D| \, d\sigma \, dt \\
&\quad + s^2 \lambda^2 \int_{Q} \phi \frac{\partial \phi}{\partial t} |\nabla \psi| ^2 |D| \, dx \, dt + \frac{1}{2} s \lambda^2 \int_{Q} \frac{\partial \phi}{\partial t} |\nabla \psi| ^2 |D| \, dx \, dt \\
&\quad - \frac{1}{2} s \lambda \int_{Q} \frac{\partial \psi}{\partial t} \Delta \psi |D| \, dx \, dt - \frac{1}{2} s \int_{Q} \frac{\partial^2 \psi}{\partial t^2} |D| \, dx \, dt \\
\text{(3.30)} &= \frac{1}{2} s \lambda \int_{\Sigma} \left| \frac{\partial \phi}{\partial t} \right| |\nabla \psi| |D| \, d\sigma \, dt + 2s^2 \lambda^2 T \int_{Q} ^3 |\nabla \psi| |D| \, dx \, dt \\
&\quad + s \lambda^2 T \int_{Q} ^3 |\nabla \psi| |D| \, dx \, dt + s^2 T \int_{Q} \phi^3 |\nabla \psi| |D| \, dx \, dt \\
&\quad + 3s T^2 \int_{Q} \phi^2 |D| \, dx \, dt + 2s \int_{Q} \phi^3 |D| \, dx \, dt \\
&\quad + 2s \gamma(\lambda) \int_{Q} \phi^3 |D| \, dx \, dt + 3s \gamma(\lambda) T^2 \int_{Q} \phi^2 |D| \, dx \, dt.
\end{align*}
\]

Now, before gathering the preceding inequalities in (3.11), we must eliminate the unsuitable surface integrals. The fact that all these integrals are multiplied by powers of \( \lambda \) having odd exponents suggests that we could get rid of them by adding our inequalities (involving \( D \)) with analogous inequalities involving a new \( \overline{D} \) defined as \( \overline{D} = e^{s\varphi} C \). It is easy to check that \( \overline{D} \) satisfies system (3.8) with \( \overline{\varphi}, \overline{\alpha}, \) and \( -\lambda \) replacing \( \varphi, \alpha, \) and \( \lambda \). So we can write
\[
\frac{\partial \overline{D}}{\partial t} + \overline{R}(t) \overline{D} - \overline{P}(t) \overline{D} = e^{s\overline{\varphi}} PG \quad \text{in } Q,
\]
where
\[
\begin{align*}
\overline{P}(t) \overline{D} &= -\Delta \overline{D} - s^2 \lambda^2 \overline{\varphi}^2 |\nabla \psi| ^2 \overline{D} - s \lambda^2 \overline{\varphi} |\nabla \psi| ^2 \overline{D} - s \lambda \overline{\varphi} \Delta \psi \overline{D} + s \frac{\partial \overline{\varphi}}{\partial t} \overline{D}, \\
\overline{R}(t) \overline{D} &= 2s \lambda \overline{\varphi} (\nabla \psi \cdot \nabla) \overline{D} - 2s \lambda^2 \overline{\varphi} |\nabla \psi| ^2 \overline{D}.
\end{align*}
\]

In the same way as before, we have
\[
2\overline{T} \leq 2\overline{J} + \int_{Q} e^{2s\varphi} |PG| ^2 \, dx \, dt,
\]
where

\[ T = - \int_{Q} \mathcal{P}(t) \mathcal{D} \cdot \mathcal{R}(t) \mathcal{D} \, dx \, dt, \quad J = \int_{Q} \frac{\partial \mathcal{D}}{\partial t} : \mathcal{P}(t) \mathcal{D} \, dx \, dt. \]

We denote by \( T_i, i = 1, \ldots, 10, \) the expressions denoting \( T \) where \( \varphi, \alpha, D, \) and \( \lambda \) are replaced by \( \bar{\varphi}, \bar{\alpha}, \bar{D}, \) and \(-\lambda\). We have \( T = \sum_{i=1}^{10} T_i \). Repeating for \( T_i \) the above computations, we obtain

\[ T_1 \geq -s \lambda^2 \sup_{\Omega} |\nabla \psi|^2 \int_{Q} \bar{\varphi} |\nabla \bar{D}|^2 \, dx \, dt \]

(3.32)

\[ -9 \max_{1 \leq j,k \leq 3} \sup_{\Omega} \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right| s \lambda \int_{Q} \bar{\varphi} |\nabla \bar{D}|^2 \, dx \, dt \]

\[ -2s \lambda \int_{\Sigma} \bar{\varphi} |\nabla \psi| \sum_{i=1}^{3} \left( \frac{\partial D_i}{\partial N} \right)^2 \, d\sigma \, dt + s \lambda \int_{\Sigma} \bar{\varphi} |\nabla \psi| |\nabla \bar{D}|^2 \, d\sigma \, dt, \]

\[ -\frac{3}{2} s \lambda^2 \int_{Q} \bar{\varphi} |\nabla \psi|^2 |\nabla \bar{D}|^2 \, dx \, dt \]

(3.33)

\[ -\max \left( 4 \sup_{\Omega} |\nabla \psi|^4, 16 \sup_{\Omega} \sum_{j,k=1}^{3} \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right|^2 \right) s \lambda^2 (\lambda^2 + 1) \]

\[ \times \int_{Q} \bar{\varphi} |\bar{D}|^2 \, dx \, dt \]

\[ -2 s^2 \lambda^3 \int_{\Sigma} \bar{\varphi} |\nabla \psi|^3 |\bar{D}|^2 \, d\sigma \, dt \]

\[ -2 s^2 \lambda^2 \int_{\Sigma} \bar{\varphi} |\nabla \psi| \sum_{i,j=1}^{3} \frac{\partial^2 \psi}{\partial x_i \partial x_j} D_i D_j d\sigma \, dt, \]

where

\[ -2 s^2 \lambda^2 \int_{\Sigma} \bar{\varphi} |\nabla \psi| \sum_{i,j=1}^{3} \frac{\partial^2 \psi}{\partial x_i \partial x_j} D_i D_j d\sigma \, dt \]

(3.34)

\[ \geq -2c \max_{1 \leq i,j \leq 3} \sup_{\partial \Omega} \left( |\nabla \psi| \sum_{j=1}^{3} \left| \frac{\partial^2 \psi}{\partial x_j} \right| \right) \]

\[ \times \left( s^{3/2} \lambda^3 \int_{Q} \bar{\varphi} |\bar{D}|^2 \, dx \, dt + s \lambda \int_{Q} \bar{\varphi} |\nabla \bar{D}|^2 \, dx \, dt \right), \]
\( c \) being the constant \( c_3 \) in (3.20). Also, we have

\[
\bar{T}_3 + \bar{T}_4 \geq \int_{\mathcal{Q}} s^3 \lambda^4 \nabla \psi^4 |\mathcal{D}|^2 \, dx \, dt
\]

(3.35)

\[
-27 \max_{1 \leq j,k \leq 3} \sup_{\Omega} \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right| \left( \max_{1 \leq j \leq 3} \sup_{\Omega} \left| \frac{\partial \psi}{\partial x_j} \right| \right)^2 s^3 \lambda^3 \int_{\mathcal{Q}} \nabla \psi^2 |\mathcal{D}|^2 \, dx \, dt
\]

\[
+ s^3 \lambda^3 \int_{\Sigma} \nabla \psi^2 |\mathcal{D}|^2 \, d\sigma \, dt ,
\]

\( \bar{T}_5 + \bar{T}_6 \geq \)

(3.36)

\[
-27 \max_{1 \leq j,k \leq 3} \sup_{\Omega} \left| \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right| \left( \max_{1 \leq j \leq 3} \sup_{\Omega} \left| \frac{\partial \psi}{\partial x_j} \right| \right)^2 s^2 \lambda^3 \int_{\mathcal{Q}} \nabla \psi^2 |\mathcal{D}|^2 \, dx \, dt
\]

\[
+ s^2 \lambda^3 \int_{\Sigma} \nabla \psi^2 |\mathcal{D}|^2 \, d\sigma \, dt ,
\]

\( \bar{T}_7 + \bar{T}_8 \geq -3 \sup_{\Omega} (\Delta \psi)^2 s^3 \lambda^3 \int_{\mathcal{Q}} \nabla \psi^3 |\mathcal{D}|^2 \, dx \, dt
\]

(3.37)

\[
- 2 \sup_{\Omega} (|\nabla \psi|^2 |\Delta \psi|) s^2 \lambda^3 \int_{\mathcal{Q}} \nabla \psi^2 |\mathcal{D}|^2 \, dx \, dt
\]

\[
- \max_{1 \leq j \leq 3} \sup_{\Omega} \left( \frac{\partial \psi}{\partial x_j} \right)^2 s \lambda \int_{\mathcal{Q}} \nabla \psi^2 |\mathcal{D}|^2 \, dx \, dt ,
\]

\( \bar{T}_9 + \bar{T}_{10} \geq -2s^2 \lambda^2 \gamma(\lambda)T \sup_{\Omega} \nabla \psi^2 \int_{\mathcal{Q}} \varphi^2 |\mathcal{D}|^2 \, dx \, dt
\]

(3.38)

\[
- 2s^2 \lambda T \sup_{\Omega} |\Delta \psi| \int_{\mathcal{Q}} \varphi^2 |\mathcal{D}|^2 \, dx \, dt
\]

\[
- 2s^2 \lambda \gamma(\lambda)T \sup_{\Omega} |\Delta \psi| \int_{\mathcal{Q}} \varphi^2 |\mathcal{D}|^2 \, dx \, dt
\]

\[
- s^2 \lambda \int_{\Sigma} \frac{\partial \varphi}{\partial t} \frac{\partial \psi}{\partial N} |\mathcal{D}|^2 \, d\sigma \, dt .
\]
Finally, we have
\[ J \leq \frac{1}{2} s \lambda \int_{\Sigma} \left| \frac{\partial \phi}{\partial t} \right| |\nabla \psi| |\overline{D}|^2 \, d\sigma \, dt + 2 s^2 \lambda^2 T \int_{Q} \phi \frac{\partial \psi}{\partial t} \left| \nabla \psi \right|^2 |\overline{D}|^2 \, dx \, dt \\
+ s \lambda^2 T \int_{Q} \phi \frac{\partial \psi}{\partial t} \left| \nabla \psi \right|^2 |\overline{D}|^2 \, dx \, dt + s \lambda T \int_{Q} \phi \frac{\partial}{\partial t} \left| \nabla \psi \right|^2 |\overline{D}|^2 \, dx \, dt \\
+ 3 s T^2 \int_{Q} \phi^2 |\overline{D}|^2 \, dx \, dt + 2 s \int_{Q} \phi^2 |\overline{D}|^2 \, dx \, dt \\
+ 2 s \gamma(\lambda) \int_{Q} \phi^2 |\overline{D}|^2 \, dx \, dt + 3 s \gamma(\lambda) T^2 \int_{Q} \phi^2 |\overline{D}|^2 \, dx \, dt. \]
\[(3.39)\]

Let us notice that
\[ \phi \leq \varphi, \quad \phi \leq \alpha, \quad |D| \leq |\overline{D}| \quad \text{in} \quad Q, \]
and
\[ \phi = \varphi, \quad \phi = \alpha, \quad \overline{D} = D \quad \text{on} \quad \Sigma. \]

Moreover, since
\[ \frac{\partial \overline{D}}{\partial x_i} = e^{i(\pi - \alpha)} \frac{\partial D}{\partial x_i} + s \left( \frac{\partial \phi}{\partial x_i} - \frac{\partial \alpha}{\partial x_i} \right) \overline{D}, \]
we also have
\[ |\nabla \overline{D}| \leq \sqrt{2} |\nabla D| + 2 \sqrt{2} s \lambda \varphi |\nabla \psi| |D| \quad \text{in} \quad Q. \]
\[(3.40)\]

Now we add inequalities (3.11) and (3.31), and then we use (3.13) through (3.16), (3.18), (3.21) through (3.28), (3.30), and (3.32) through (3.39). (Remember that \( I = \sum_{i=1}^{10} I_i \) and \( \overline{T} = \sum_{i=1}^{10} \overline{T}_i \).) Taking (3.40) through (3.42) into account, we obtain for \( s > 1, \lambda > 1, \)
\[ s \lambda^2 \int_{Q} \phi |\nabla \psi|^2 |\nabla D|^2 \, dx \, dt + s^3 \lambda^4 \int_{Q} \phi^3 |\nabla \psi|^4 |D|^2 \, dx \, dt \\
\leq c \left( s \lambda \int_{Q} \phi |\nabla D|^2 \, dx \, dt + s^3 \lambda^3 \int_{Q} \phi^3 |D|^2 \, dx \, dt \\
+ s^2 \lambda^4 \gamma(\lambda) \int_{Q} \phi^3 |D|^2 \, dx \, dt + \int_{Q} e^{2 \alpha} |PG|^2 \, dx \, dt \right), \]
\[(3.43)\]
where the constant \( c \) depends only on \( |\psi|_{C^2(\overline{Q})} \) and \( T \). (Here the sum of the four surface integrals in (3.13) and (3.32) has been calculated by using the boundary
conditions in form (3.17); its final expression is

\[ -4s^2 \lambda^2 \int_{\Sigma} \varphi^2 |\nabla \psi| \sum_{i,j=1}^{3} \frac{\partial^2 \psi}{\partial x_i \partial x_j} D_i D_j \, d\sigma \, dt \]

and can be treated in the same manner as the surface integral in (3.19) by taking \( \alpha = \frac{1}{2} \) and \( \beta = \frac{1}{4} \) in (3.20) where \( s \) is replaced by \( s^2 (\varphi^*(t))^2 \).

By (3.3), there exists \( \rho_0 > 0 \) such that \( |\nabla \psi| \geq \rho_0 \) in \( \Omega \setminus \omega_0 \), so we can rewrite inequality (3.43) as

\[
\rho_0^2 s \lambda^2 \int_{Q \setminus Q_{\omega_0}} \varphi |\nabla D|^2 \, dx \, dt + \rho_0^4 s^3 \lambda^4 \int_{Q \setminus Q_{\omega_0}} \varphi^3 |D|^2 \, dx \, dt \leq c \left( s \lambda \int_{Q \setminus Q_{\omega_0}} \varphi |\nabla D|^2 \, dx \, dt + s^3 \lambda^3 \int_{Q} \varphi^3 |D|^2 \, dx \, dt \right. \\
+ s^3 \lambda^4 \gamma(\lambda) \int_{Q} \varphi^3 |D|^2 \, dx \, dt + \int_{Q} e^{2s\alpha} |G|^2 \, dx \, dt \right).
\]

(3.44)

This inequality is the key to the proof, which allows us to replace integration over \( Q \) by integration over \( Q_{\omega_0} \) in the right-hand side by using only the fact that the similar terms in the left-hand side contain powers of \( s \) and \( \lambda \) with greater exponents. Indeed, taking

\[ \lambda > \lambda_0 = \max \left( \frac{c + 1}{\rho_0^2}, \frac{c + 1}{\rho_0^4}, 1 \right), \]

we have \( \rho_0^2 s \lambda^2 - cs \lambda > s \lambda \) and \( \rho_0^4 \lambda - c - 1 > 0 \), where \( c \) is the constant in (3.44). Then taking

\[ s > s_0(\lambda) = \max \left( \frac{c \lambda \gamma(\lambda)}{\rho_0^4 \lambda - c - 1}, 1 \right), \]

we have \( \rho_0^4 s^3 \lambda^4 - cs^3 \lambda^3 - cs^2 \lambda^4 \gamma(\lambda) > s^3 \lambda^3 \), and consequently

\[
s^3 \lambda^3 \int_{Q \setminus Q_{\omega_0}} \varphi^3 |D|^2 \, dx \, dt + s \lambda \int_{Q \setminus Q_{\omega_0}} \varphi |\nabla D|^2 \, dx \, dt \leq \\
(1 + \lambda \gamma(\lambda)) s^3 \lambda^3 \int_{Q_{\omega_0}} \varphi^3 |D|^2 \, dx \, dt + s \lambda \int_{Q_{\omega_0}} \varphi |\nabla D|^2 \, dx \, dt + \int_{Q} e^{2s\alpha} |PG|^2 \, dx \, dt.
\]
Adding the two integrals from the left-hand side of the above inequality but taken over $Q_{\omega_0}$ to both sides, we obtain

$$
\begin{align*}
\lambda^3 \int_{Q} \varphi^3 |D|^2 \, dx \, dt + s \lambda \int_{Q} \varphi |\nabla D|^2 \, dx \, dt \leq \\
c(\lambda) \left( \lambda^3 \int_{Q_{\omega_0}} \varphi^3 |D|^2 \, dx \, dt + s \int_{Q_{\omega_0}} \varphi |\nabla D|^2 \, dx \, dt + \int_{Q} e^{2s\lambda} |PG|^2 \, dx \, dt \right)
\end{align*}
$$

(3.45)

for $\lambda > \lambda_0$ and $s > s_0(\lambda)$, where $c(\lambda) = 2\lambda^3 (1 + \gamma(\lambda))$.

Now we come back to $C$ ($D = e^{s\alpha} C$). After some calculation, inequality (3.45) becomes

$$
\begin{align*}
\lambda^3 \int_{Q} e^{2s\alpha} \varphi^3 |C|^2 \, dx \, dt + s \lambda \int_{Q} e^{2s\alpha} \varphi |\nabla C|^2 \, dx \, dt \leq \\
c(\lambda) \left( \lambda^3 \int_{Q_{\omega_0}} e^{2s\alpha} \varphi^3 |C|^2 \, dx \, dt + s \int_{Q_{\omega_0}} e^{2s\alpha} \varphi |\nabla C|^2 \, dx \, dt + \int_{Q} e^{2s\alpha} |PG|^2 \, dx \, dt \right)
\end{align*}
$$

(3.46)

for $\lambda > \lambda_0$ and $s > s_0(\lambda)$, where $\lambda_0$ is possibly greater than $\lambda_0$ before in order to satisfy $\lambda_0 \geq 2 \sup_{\Omega} |\nabla \psi|^2 + \frac{1}{\gamma}$, and $c(\lambda)$ is the constant $c(\lambda)$ before but multiplied by $2 \max(1 + \lambda(1 + \lambda) \sup_{\Omega} |\nabla \psi|^2, 1 + \lambda)$.

Now we remove the integral of $|\nabla C|^2$ over $Q_{\omega_0}$ from the right-hand side of (3.46). To this aim, choose $\chi \in C^\infty_0(\Omega)$ such that $\chi = 1$ in $\overline{\omega_0}$ and $\chi = 0$ in $\Omega \setminus \omega_1$. (We note that $\omega_0 \subseteq \omega_1$.) We multiply equation (3.2) by $\chi \varphi e^{2s\alpha} C$ and integrate over $Q$. Let us examine the terms of the equation obtained in this way.

We have

$$
\begin{align*}
\int_{Q} \chi \varphi e^{2s\alpha} C \cdot \frac{\partial C}{\partial t} \, dx \, dt \\
= -\frac{1}{2} \int_{Q} \frac{\partial}{\partial t} (\chi \varphi e^{2s\alpha}) |C|^2 \, dx \, dt \\
\leq \left( \frac{T^4}{2^4} + 2s(1 + \gamma(\lambda)) \right) \frac{T^3}{2^2} \int_{Q_{\omega_1}} e^{2s\alpha} \varphi^3 |C|^2 \, dx \, dt .
\end{align*}
$$

(3.47)
Using Green’s formula, we obtain

\[
\int_Q \chi \phi e^{2s\alpha} C \cdot \Delta C \, dx \, dt
\]

\[
= - \sum_{i=1}^{3} \int_Q \phi e^{2s\alpha} C_i \nabla \chi \cdot \nabla C_i \, dx \, dt
\]

\[
(3.48)
\]

\[
- \lambda \sum_{i=1}^{3} \int_Q \chi \phi e^{2s\alpha} C_i \nabla \psi \cdot \nabla C_i \, dx \, dt
\]

\[
- 2s\lambda \sum_{i=1}^{3} \int_Q \chi \phi^2 e^{2s\alpha} \nabla \psi \cdot \nabla C_i \, dx \, dt - \int_Q \chi \phi e^{2s\alpha} |\nabla C|^2 \, dx \, dt.
\]

We also have

\[
- \sum_{i=1}^{3} \int_Q \phi e^{2s\alpha} C_i \nabla \chi \cdot \nabla C_i \, dx \, dt
\]

\[
(3.49)
\]

\[
\leq \sup_{\Omega_{\psi}} |\nabla \chi| \int_{\Omega_{\psi}} e^{2s\alpha} \phi \left( \frac{\lambda}{2} |C|^2 + \frac{1}{2\lambda} |\nabla C|^2 \right) \, dx \, dt,
\]

\[
- \lambda \sum_{i=1}^{3} \int_Q \chi \phi e^{2s\alpha} C_i \nabla \psi \cdot \nabla C_i \, dx \, dt
\]

\[
(3.50)
\]

\[
\leq \sup_{\Omega_{\psi}} |\nabla \psi| \int_{\Omega_{\psi}} e^{2s\alpha} \phi \left( \frac{\lambda^3}{2} |C|^2 + \frac{1}{2\lambda} |\nabla C|^2 \right) \, dx \, dt,
\]

\[
- 2s\lambda \sum_{i=1}^{3} \int_Q \chi \phi^2 e^{2s\alpha} C_i \nabla \psi \cdot \nabla C_i \, dx \, dt
\]

\[
(3.51)
\]

\[
\leq \sup_{\Omega_{\psi}} |\nabla \psi| \int_{\Omega_{\psi}} e^{2s\alpha} \left( s^2 \phi^3 |C|^2 + \frac{1}{\lambda} |\nabla C|^2 \right) \, dx \, dt,
\]

\[
(3.52)
\]

\[
- \int_Q \chi \phi e^{2s\alpha} C \cdot PG \, dx \, dt \leq \int_{\Omega_{\psi}} e^{2s\alpha} \left( \frac{s}{2} \phi^2 |C|^2 + \frac{1}{2s} |PG|^2 \right) \, dx \, dt.
\]
Taking equation (3.48) and inequalities (3.47) and (3.49) through (3.52) together, we obtain
\[
\int_Q e^{2s\alpha} \phi |\nabla C|^2 \, dx \, dt \leq \nonumber
\]
\[
(3.53) \nonumber
\]
\[
c(\lambda) \left( \int_{Q_{\omega_1}} e^{2s\alpha} s^2 \phi^3 |C|^2 \, dx \, dt + \int_Q e^{2s\alpha} \frac{1}{s} |P G|^2 \, dx \, dt \right)
\]
for \( \lambda > \frac{1}{2} \sup_{\Omega} |\nabla \chi| + \frac{3}{2} \sup_{\Omega} |\nabla \psi|, s > 1. \) (Here we have also used the obvious inequalities \( \phi \leq (T/2)^{T/4} \phi^3 \) and \( \phi^2 \leq (T/2)^{T/3}. \) ) Inserting inequality (3.53) into (3.46), we have
\[
\int_Q e^{2s\alpha} \phi^3 |C|^2 \, dx \, dt + s \lambda \int_Q e^{2s\alpha} \phi |\nabla C|^2 \, dx \, dt \leq \nonumber
\]
\[
(3.54) \nonumber
\]
\[
c(\lambda) \left( \int_{Q_{\omega_1}} e^{2s\alpha} s^2 \phi^3 |C|^2 \, dx \, dt + \int_Q e^{2s\alpha} |P G|^2 \, dx \, dt \right)
\]
for \( \lambda > \lambda_0 \) and \( s > s_0(\lambda), \) where \( \lambda_0 \) is chosen as in (3.46) but also satisfying \( \lambda_0 \geq \frac{1}{2} \sup_{\Omega} |\nabla \chi| + \frac{3}{2} \sup_{\Omega} |\nabla \psi| \).

Now we shall estimate the integrals of \( e^{2s\alpha} \phi^{-1} |\partial C/\partial t|^2 \) and \( e^{2s\alpha} \phi^{-1} |\Delta C|^2 \) over \( Q \) by means of the integral of \( e^{2s\alpha} \phi |\nabla C|^2 \). We first multiply equation (3.2) by \( e^{2s\alpha} \phi^{-1}(\partial C/\partial t) \) and integrate over \( Q \). Integrating successively by parts with respect to \( x \) and \( t \) and taking the second boundary condition in (3.2) into account, we obtain that
\[
\int_Q e^{2s\alpha} \phi^{-1} \frac{\partial C}{\partial t} \cdot \nabla C \, dx \, dt
\]
\[
= -\int_Q \nabla \left( e^{2s\alpha} \phi^{-1} \frac{\partial C}{\partial t} \right) \cdot \nabla C \, dx \, dt + \int_\Sigma \left( e^{2s\alpha} \phi^{-1} \frac{\partial C}{\partial t} \times \nabla C \right) \cdot N \, d\sigma \, dt
\]
\[
= \frac{1}{2} \int_Q e^{2s\alpha} \phi^{-1} \frac{\partial}{\partial t} \left| \nabla C \right|^2 \, dx \, dt + \int_Q \left( \frac{\partial C}{\partial t} \times \nabla (e^{2s\alpha} \phi^{-1}) \right) \cdot \nabla C \, dx \, dt
\]
\[
+ \int_\Sigma e^{2s\alpha} \phi^{-1} (N \times \nabla C) \cdot \frac{\partial C}{\partial t} \, d\sigma \, dt
\]
\[
= \frac{1}{2} \int_Q e^{2s\alpha} \left( \phi^{-1} \frac{\partial \phi}{\partial t} - 2s \phi^{-1} \frac{\partial \phi}{\partial t} \right) \left| \nabla C \right|^2 \, dx \, dt
\]
\[
+ \lambda \int_Q \left( \nabla C \times \frac{\partial C}{\partial t} \right) \cdot e^{2s\alpha} \left( -\phi^{-1} + 2s \nabla \psi \right) \, dx \, dt.
\]
We take this equality together with the following inequalities:

\[
\frac{1}{2} \int_{Q} e^{2s} \phi^2 \frac{\partial \phi}{\partial t} |\text{curl } C|^2 \, dx \, dt \leq \frac{T^7}{25} \int_{Q} e^{2s} \phi |\nabla C|^2 \, dx \, dt ,
\]

\[-s \int_{Q} e^{2s} \phi^{-1} \frac{\partial \phi}{\partial t} |\text{curl } C|^2 \, dx \, dt \leq T^3 (1 + \gamma (\lambda)) s \int_{Q} e^{2s} \phi |\nabla C|^2 \, dx \, dt ,
\]

\[\lambda \int_{Q} e^{2s} \phi^{-1} \nabla \psi \cdot \left( \text{curl } C \times \frac{\partial C}{\partial t} \right) \, dx \, dt \leq \]

\[\frac{3}{2^8} \left( \sup_{\Omega} |\nabla \psi| \right)^2 \lambda^2 \int_{Q} e^{2s} \phi |\nabla C|^2 \, dx \, dt + \frac{1}{6} \int_{Q} e^{2s} \phi^{-1} \left| \frac{\partial C}{\partial t} \right|^2 \, dx \, dt ,
\]

\[-2s \lambda \int_{Q} e^{2s} \nabla \psi \cdot \left( \text{curl } C \times \frac{\partial C}{\partial t} \right) \, dx \, dt \leq \]

\[12 \left( \sup_{\Omega} |\nabla \psi| \right)^2 s^2 \lambda^2 \int_{Q} e^{2s} \phi |\nabla C|^2 \, dx \, dt + \frac{1}{6} \int_{Q} e^{2s} \phi^{-1} \left| \frac{\partial C}{\partial t} \right|^2 \, dx \, dt ,
\]

\[\int_{Q} e^{2s} \phi^{-1} \frac{\partial C}{\partial t} \cdot PG \, dx \, dt \leq \]

\[\frac{1}{6} \int_{Q} e^{2s} \phi^{-1} \left| \frac{\partial C}{\partial t} \right|^2 \, dx \, dt + \frac{3}{2} T^4 \int_{Q} e^{2s} |PG|^2 \, dx \, dt .
\]

(Here we have used the obvious inequalities $|\text{curl } C|^2 \leq 2 |\nabla C|^2$, $\phi^{-1/2} \leq T^2 / 2^2$, $\phi^{1/2} \leq (T^2 / 2^2) \phi$, and $\phi^{-1} \leq (T^8 / 2^8) \phi$.) These yield

\[\int_{Q} e^{2s} (s \phi)^{-1} \left| \frac{\partial C}{\partial t} \right|^2 \, dx \, dt \leq \]

\[c(\lambda) \left( s \int_{Q} e^{2s} \phi |\nabla C|^2 \, dx \, dt + \int_{Q} e^{2s} |PG|^2 \, dx \, dt \right) \text{ for } s \geq 1 .
\]

Using (3.54) too, we obtain

\[\int_{Q} e^{2s} (s \phi)^{-1} \left| \frac{\partial C}{\partial t} \right|^2 \, dx \, dt \leq \]

\[c(\lambda) \left( \int_{Q_{=1}} e^{2s} s^3 \phi^3 |\nabla C|^2 \, dx \, dt + \int_{Q} e^{2s} |PG|^2 \, dx \, dt \right) \text{ (3.55)}
\]
for $\lambda > \lambda_0$ and $s > s_0(\lambda)$. Then we multiply equation (3.2) by $e^{2s\alpha} \varphi^{-1} \Delta C$ and integrate over $Q$. Since

$$- \int_{Q} e^{2s\alpha} \varphi^{-1} \Delta C \cdot \frac{\partial C}{\partial t} \, dx \, dt \leq \frac{1}{4} \int_{Q} e^{2s\alpha} \varphi^{-1} |\Delta C|^2 \, dx \, dt + \int_{Q} e^{2s\alpha} \varphi^{-1} \left| \frac{\partial C}{\partial t} \right|^2 \, dx \, dt$$

and

$$\int_{Q} e^{2s\alpha} \varphi^{-1} \Delta C \cdot PG \, dx \, dt \leq \frac{1}{4} \int_{Q} e^{2s\alpha} \varphi^{-1} |\Delta C|^2 \, dx \, dt + \int_{Q} e^{2s\alpha} \varphi^{-1} |PG|^2 \, dx \, dt,$$

we have

$$\frac{1}{2} \int_{Q} e^{2s\alpha} \varphi^{-1} |\Delta C|^2 \, dx \, dt \leq \int_{Q} e^{2s\alpha} \varphi^{-1} \left| \frac{\partial C}{\partial t} \right|^2 \, dx \, dt + \frac{T^4}{2 \pi} \int_{Q} e^{2s\alpha} |PG|^2 \, dx \, dt.$$

This inequality together with (3.55) yields

$$\int_{Q} e^{2s\alpha} (s\varphi)^{-1} |\Delta C|^2 \, dx \, dt \leq c(\lambda) \left( \int_{Q_{=1}} e^{2s\alpha} s^3 \varphi^3 |C|^2 \, dx \, dt + \int_{Q} e^{2s\alpha} |PG|^2 \, dx \, dt \right)$$

(3.56) for $\lambda > \lambda_0$ and $s > s_0(\lambda)$.

Finally, let us show that we have an estimate similar to (3.56) also for

$$\sum_{i,j=1}^{3} \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2.$$

To this end, we set $\tilde{C} = e^{s\alpha} (s\varphi)^{-1/2} C$. By a well-known a priori estimate for the solutions of the Poisson equation, we have

$$|\tilde{C}(\cdot, t)|_{(H^3(\Omega))^3} \leq c \left( |\Delta \tilde{C}(\cdot, t)|_{(L^2(\Omega))^3} + |\tilde{C}(\cdot, t)|_{(H^{3/2}(\partial \Omega))^3} \right) = c \left( |\Delta \tilde{C}(\cdot, t)|_{(L^2(\Omega))^3} + e^{s\alpha(t)} (s\varphi^*(t))^{-1/2} |C(\cdot, t)|_{(H^{3/2}(\partial \Omega))^3} \right).$$

But the trace theorem and a well-known regularity result for the stationary Stokes equations with the boundary conditions in (3.2) (see, for instance, [19, inequality (2.8)]) show that

$$|C(\cdot, t)|_{(H^{3/2}(\partial \Omega))^3} \leq c_1 |C(\cdot, t)|_{(H^2(\Omega))^3} \leq c_2 |\Delta C(\cdot, t)|_{(H^2(\Omega))^3}.$$

Consequently,

$$|\tilde{C}(\cdot, t)|_{(H^2(\Omega))^3} \leq c \left( |\Delta \tilde{C}(\cdot, t)|_{(L^2(\Omega))^3} + e^{s\alpha(t)} (s\varphi^*(t))^{-1/2} |\Delta C(\cdot, t)|_{(L^2(\Omega))^3} \right) \leq c \left( |\Delta \tilde{C}(\cdot, t)|_{(L^2(\Omega))^3} + |e^{s\alpha} (s\varphi)^{-1/2} |\Delta C(\cdot, t)|_{(L^2(\Omega))^3}| \right).$$
(It is easy to check that \( e^{s\alpha^* (t)} (s\varphi^*(t))^{-1/2} \leq e^{s\alpha} (s\varphi)^{-1/2} \). So, we have
\[
\int Q \sum_{i,j=1}^3 \left| \frac{\partial^2 \widetilde{C}}{\partial x_i \partial x_j} \right|^2 \, dx \, dt \leq c \left( \int Q |\Delta \widetilde{C}|^2 \, dx \, dt + \int Q e^{2s\alpha} (s\varphi)^{-1} |\Delta C|^2 \, dx \, dt \right).
\]

A tedious but simple calculation gives the following inequalities:
\[
\sum_{i,j=1}^3 \left| \frac{\partial^2 \widetilde{C}}{\partial x_i \partial x_j} \right|^2 \geq \frac{1}{2} e^{2s\alpha} (s\varphi)^{-1} \sum_{i,j=1}^3 \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 - e^{2s\alpha} \lambda^4 \left( 8s^3 \varphi^3 + 4s \varphi + \frac{1}{2} s^{-1} \varphi^{-1} \right) |\nabla \psi|^4 |C|^2
\]
\[
- e^{2s\alpha} \lambda^2 (8s \varphi - 8 + 2s^{-1} \varphi^{-1}) \sum_{i,j=1}^3 \left| \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right|^2 |C|^2
\]
\[
- 2e^{2s\alpha} \lambda^2 (8s \varphi - 8 + 2s^{-1} \varphi^{-1}) |\nabla \psi|^2 |\nabla C|^2,
\]
\[
|\Delta \widetilde{C}|^2 \leq 2e^{2s\alpha} (s\varphi)^{-1} |\Delta C|^2
\]
\[
+ e^{2s\alpha} \lambda^4 \left( 8s^3 \varphi^3 + 4s \varphi + \frac{1}{2} s^{-1} \varphi^{-1} \right) |\nabla \psi|^4 |C|^2
\]
\[
+ e^{2s\alpha} \lambda^2 (8s \varphi - 8 + 2s^{-1} \varphi^{-1}) \sum_{i=1}^3 \left| \frac{\partial^2 \psi}{\partial x_i^2} \right|^2 |C|^2
\]
\[
+ 2e^{2s\alpha} \lambda^2 (8s \varphi - 8 + 2s^{-1} \varphi^{-1}) |\nabla \psi|^2 |\nabla C|^2.
\]

Inserting them into the preceding inequality, we obtain
\[
\int Q e^{2s\alpha} (s\varphi)^{-1} \sum_{i,j=1}^3 \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \, dx \, dt \leq c(\lambda) \left( \int Q e^{2s\alpha} s^3 \varphi^3 |C|^2 \, dx \, dt + \int Q e^{2s\alpha} s\varphi |\nabla C|^2 \, dx \, dt + \int Q e^{2s\alpha} (s\varphi)^{-1} |\Delta C|^2 \, dx \, dt \right).
\]
This together with inequalities (3.54) and (3.56) gives
\[
\int_{Q} e^{2s\alpha} (s\varphi)^{-1} \sum_{i,j=1}^{3} \left| \frac{\partial^{2} C}{\partial x_{i} \partial x_{j}} \right|^{2} dx \ dt \leq \nonumber
\]
\[
c(\lambda) \left( \int_{Q} e^{2s\alpha} s^{3} \varphi^{3} |C|^{2} dx \ dt + \int_{Q} e^{2s\alpha} |PG|^{2} dx \ dt \right) \nonumber
\]
(3.57)
for \( \lambda > \lambda_{0} \) and \( s > s_{0}(\lambda) \). Taking inequalities (3.54), (3.55), and (3.57) together, we have
\[
\int_{Q} e^{2s\alpha} \left( \frac{1}{s\varphi} \left| \frac{\partial C}{\partial t} \right|^{2} + \sum_{i,j=1}^{3} \left| \frac{\partial^{2} C}{\partial x_{i} \partial x_{j}} \right|^{2} \right) dx \ dt + s\lambda \varphi |\nabla C|^{2} + s^{3}\lambda^{3} \varphi^{3} |C|^{2} \nonumber
\]
\[
\leq c(\lambda) \left( \int_{Q} e^{2s\alpha} |PG|^{2} dx \ dt + \int_{Q} e^{2s\alpha} s^{3} \varphi^{3} |C|^{2} dx \ dt \right) \nonumber
\]
(3.58)
for \( \lambda > \lambda_{0} \) and \( s > s_{0}(\lambda) \).

It remains to eliminate \( P \) in the right-hand side of (3.58). We may write \( G \) as
\[
G = PG + \nabla q \nonumber
\]
(3.59)
for some function \( q \). Applying the divergence operator to both sides of the above equality and then multiplying it by \( N \) on \( \Sigma \), we obtain
\[
\Delta q = \text{div} \ G \quad \text{in} \ Q , \quad \left| \frac{\partial q}{\partial N} \right| \leq c |C| \quad \text{on} \ \Sigma , \nonumber
\]
where \( c \) is the constant in the statement of Theorem 3.2. Now treating \( q \) in the same manner as \( C \), we obtain (after careful calculations) that \( q \) satisfies the following Carleman-type inequality (see [2, theorem 1.2.3] for a similar situation):
\[
\int_{Q} e^{2s\alpha} s\varphi |\nabla q|^{2} dx \ dt \nonumber
\]
\[
\leq c(\lambda) \left( \int_{Q} e^{2s\alpha} |\text{div} \ G|^{2} dx \ dt + \int_{Q} e^{2s\alpha} s^{3} \varphi^{3} q^{2} dx \ dt \right) \nonumber
\]
\[
+ \frac{1}{2} s^{3}\lambda^{3} \int_{Q} e^{2s\alpha} \varphi^{3} |C|^{2} dx \ dt + \frac{1}{2} s\lambda \int_{Q} e^{2s\alpha} \varphi |\nabla C|^{2} dx \ dt \nonumber
\]
(3.60)
for \( \lambda > \lambda_{0} \) and \( s > s_{0}(\lambda) \) (where \( \lambda_{0} \) and \( s_{0}(\lambda) \) are possibly larger than those before). Then, some considerations involving equations (3.2) yield the following
So let us study the controllability of the null solution of system (2.5) where

\[
\text{(3.61)}
\]

for some positive constants c and z. Finally, inserting (3.60) and (3.61) into (3.58) via (3.59), we obtain (3.7).

\[
4 \text{ Proof of Theorem 2.1}
\]

We shall reduce the local exact controllability of the null solution of system (2.4) to the global exact controllability of the null solution of the linear system (2.5) by using the infinite-dimensional Kakutani’s fixed-point theorem on the convex compact set \(K \subset (L^2(Q))^3\) defined by

\[
K = \left\{ (w, E) \in (L^\infty(0, T; D(A_1)) \cap H^1(0, T; V_1)) \times (L^\infty(0, T; D(A_2)) \cap H^1(0, T; V_2)) : \mu(w, E) \leq M \right\},
\]

where

\[
\mu(w, E) = \|w\|_{L^\infty(0, T; D(A_1))} + \|w\|_{H^1(0, T; V_1)} + \|E\|_{L^\infty(0, T; D(A_2))} + \|E\|_{H^1(0, T; V_2)}.
\]

So let us study the controllability of the null solution of system (2.5).

**Lemma 4.1** Let \(\Omega\) and \(\omega\) be as in the statement of Theorem 2.1. Then there is \(M > 0\) such that for all \((w, E) \in K\) and \((y^0, B^0) \in D(A_1) \times D(A_2)\) there exist \(u, v \in H^1(0, T; (L^2(\Omega))^3)\) and \((y, B, p) \in (L^\infty(0, T; D(A_1)) \cap H^1(0, T; V_1)) \times (L^\infty(0, T; D(A_2)) \cap H^1(0, T; V_2)) \times H^1(0, T; \Omega))\) satisfying (2.5) such that

\[
(4.1) \quad y(x, T) = 0, \quad B(x, T) = 0 \text{ a.e. } x \in \Omega, \\
|u|_{H^1(0, T; (L^2(\Omega))^3)} + |v|_{H^1(0, T; (L^2(\Omega))^3)} \leq \\
\beta(M^2 + 1)^2 e^{c(M^2+1)} (|y^0|_{(L^2(\Omega))^3}^2 + |B^0|_{(L^2(\Omega))^3}^2),
\]

for some positive constants \(c\) and \(\beta\) independent of \(w, E, y^0, B^0, \text{ and } M\).

**Proof:** For \(\varepsilon > 0\) we consider the following optimal control problem: Minimize

\[
\left(\mathcal{P}_\varepsilon\right) \quad \frac{1}{2} \int_Q e^{-2\varepsilon\delta} (|u|^2 + |v|^2) dx \, dt + \frac{1}{2\varepsilon} \int_\Omega (|y(x, T)|^2 + |B(x, T)|^2) dx
\]
over all $u, v \in (L^2(Q))^3$, where $y$ and $B$ satisfy (2.5). Here $s, \alpha$, and $\delta$ are chosen as in Theorems 3.1 and 3.2. For each $\varepsilon > 0$, problem $(P_\varepsilon)$ has a unique solution $(u_\varepsilon, v_\varepsilon, y_\varepsilon, B_\varepsilon, p_\varepsilon)$. We shall show that the solution $(u, v, y, B, p)$ of the null controllability problem for the linear system (2.5) (which we are looking for) is a limit “point” (in a certain topology) of the sequence $(u_\varepsilon, v_\varepsilon, y_\varepsilon, B_\varepsilon, p_\varepsilon)$ as $\varepsilon \to 0$. To this aim, we need certain estimates for $u_\varepsilon, v_\varepsilon, y_\varepsilon, B_\varepsilon$, and $p_\varepsilon$. The main effort of this proof is focused on obtaining these estimates.

By the Pontryagin maximum principle, the solution $(u_\varepsilon, v_\varepsilon, y_\varepsilon, B_\varepsilon, p_\varepsilon)$ of problem $(P_\varepsilon)$ satisfies the following necessary conditions:

\begin{equation}
 u_\varepsilon = me^{2\varepsilon \alpha \delta} z_\varepsilon, \quad v_\varepsilon = me^{2\varepsilon \alpha \delta} C_\varepsilon \text{ a.e. in } Q, \tag{4.3}
\end{equation}

where $(z_\varepsilon, C_\varepsilon, q_\varepsilon)$ is the solution of the backward dual system of (2.5):

\begin{align*}
 &\frac{\partial z_\varepsilon}{\partial t} + \Delta z_\varepsilon + ((w + y_\varepsilon) \cdot \nabla) z_\varepsilon - z_\varepsilon \cdot (\nabla y_\varepsilon) \\
 &\quad - ((E + B_\varepsilon) \cdot \nabla) C_\varepsilon - C_\varepsilon \cdot (\nabla B_\varepsilon) + \nabla q_\varepsilon = 0 \quad \text{in } Q, \\
 &\frac{\partial C_\varepsilon}{\partial t} - \text{curl(curl } C_\varepsilon) + P((w + y_\varepsilon) \cdot \nabla) C_\varepsilon + C_\varepsilon \cdot (\nabla y_\varepsilon) \\
 &\quad - ((E + B_\varepsilon) \cdot \nabla) z_\varepsilon + z_\varepsilon \cdot (\nabla B_\varepsilon) + (z_\varepsilon \cdot \nabla) E = 0 \quad \text{in } Q, \\
 &\quad \nabla \cdot z_\varepsilon = 0, \quad \nabla \cdot C_\varepsilon = 0 \quad \text{in } Q, \\
 &\quad z_\varepsilon = 0, \quad C_\varepsilon \cdot N = 0, \quad (\text{curl } C_\varepsilon) \times N = 0 \quad \text{on } \Sigma, \\
 &\quad z_\varepsilon(\cdot, T) = -\frac{1}{\varepsilon} y_\varepsilon(\cdot, T), \quad C_\varepsilon(\cdot, T) = -\frac{1}{\varepsilon} B_\varepsilon(\cdot, T) \quad \text{in } \Omega, \tag{4.4}
\end{align*}

where the products $z_\varepsilon \cdot (\nabla y_\varepsilon), C_\varepsilon \cdot (\nabla B_\varepsilon), C_\varepsilon \cdot (\nabla y_\varepsilon), \text{ and } z_\varepsilon \cdot (\nabla B_\varepsilon)$ are defined in the same manner as the product $E \cdot (\nabla B)$ in (2.5). Using (2.5), (4.3), and (4.4), we obtain

\begin{align*}
 &\int_{Q_\omega} e^{2\alpha \delta} (|z_\varepsilon|^2 + |C_\varepsilon|^2) dx \, dt + \frac{1}{\varepsilon} (|y_\varepsilon(T)|^2_{L^2} + |C_\varepsilon(T)|^2_{L^2}) \\
 &\quad = -\int_{\Omega} z_\varepsilon(x, 0) \cdot y_0(x) dx - \int_{\Omega} C_\varepsilon(x, 0) \cdot B_0(x) dx \\
 &\quad \leq |z_\varepsilon(0)|_{L^2} |y_0|_{L^2} + |C_\varepsilon(0)|_{L^2} |B_0|_{L^2}. \tag{4.5}
\end{align*}

This is the first needed inequality.
In what follows we shall simply write \((z, C, q)\) instead of \((z_e, C_e, q_e)\). Let us rewrite system (4.4) as
\[
\frac{\partial z}{\partial t} + \Delta z = g - \nabla q \quad \text{in } Q, \\
\frac{\partial C}{\partial t} + \Delta C = PG \quad \text{in } Q, \\
\nabla \cdot z = 0, \quad \nabla \cdot C = 0 \quad \text{in } Q,
\]
(4.6)
\[
z = 0, \quad C \cdot N = 0, \quad (\text{curl } C) \times N = 0 \quad \text{on } \Sigma, \\
z(\cdot, T) = \frac{-1}{\varepsilon} y_e(\cdot, T), \quad C(\cdot, T) = \frac{-1}{\varepsilon} B_e(\cdot, T) \quad \text{in } \Omega,
\]
where
\[
g = -((w + y_e) \cdot \nabla)z + z \cdot (\nabla y_e) + ((E + B_e) \cdot \nabla)C + C \cdot (\nabla B_e), \\
G = -((w + y_e) \cdot \nabla)C - C \cdot (\nabla y_e) + ((E + B_e) \cdot \nabla)z - z \cdot (\nabla B_e) - (z \cdot \nabla)E.
\]
Because \(z, C,\) and \(E\) are divergence-free, we have
\[
\nabla \cdot g = - \sum_{i,j=1}^{3} \frac{\partial(w + y_e)_i}{\partial x_j} \frac{\partial z_j}{\partial x_i} + \nabla z \cdot \nabla y_e + z \cdot \Delta y_e \\
+ \sum_{i,j=1}^{3} \frac{\partial(E + B_e)_i}{\partial x_j} \frac{\partial C_j}{\partial x_i} + \nabla C \cdot \nabla B_e + C \cdot \Delta B_e, \\
\nabla \cdot G = - \sum_{i,j=1}^{3} \frac{\partial(w + y_e)_i}{\partial x_j} \frac{\partial C_j}{\partial x_i} - \nabla C \cdot \nabla y_e - C \cdot \Delta y_e \\
+ \sum_{i,j=1}^{3} \frac{\partial B_{ei}}{\partial x_j} \frac{\partial z_j}{\partial x_i} - \nabla z \cdot \nabla B_e - z \cdot \Delta B_e.
\]
Since \(y_e, B_e \in (H^3(\Omega))^3,\) by the Sobolev imbedding theorem we have \(|y_e|,|B_e|,|\nabla y_e|,\) and \(|\nabla B_e| \in L^\infty(\Omega).\) Then a.e. in \(Q,\)
\[
|g| \leq c\left((|w| + 1)|\nabla z| + (|E| + 1)|\nabla C| + |z| + |C|\right), \\
|\nabla \cdot g| \leq c\left((|\nabla w| + 1)|\nabla z| + (|\nabla E| + 1)|\nabla C| + |\Delta y_e||z| + |\Delta B_e||C|\right), \\
|G| \leq c\left((|w| + 1)|\nabla C| + (|E| + 1)|\nabla z| + |C| + |z| + |z||\nabla E|\right), \\
|\nabla \cdot G| \leq c\left((|\nabla w| + 1)|\nabla C| + |\nabla z| + |\Delta y_e||C| + |\Delta B_e||z|\right).
\]
(Here and in the following, \(c\) denotes positive constants. The dependence of \(c\) on the parameters \(\lambda, s,\) and \(M\) is expressed by writing \(c(\lambda), c(s),\) and \(c(M).\))
Because \( y \in (H^3(\Omega))^3 \), by the trace theorem and the Sobolev imbedding theorem (for fractional order \( \frac{3}{2} \) and dimension 2), we also have \( |\nabla y_e| \in L^\infty(\partial \Omega) \). But this assures that \( G \cdot N \) satisfies the boundary inequality in the statement of Theorem 3.2. (Some calculation is needed to verify the last assertion.) Coupling the Carleman inequalities (3.6) and (3.7) for system (4.6) with \( g \) and \( G \) given before and then using (4.7) through (4.10), we obtain

\[
\int_\Omega e^{2s\alpha} \left( \frac{1}{s\varphi} \left( \left| \frac{\partial z}{\partial t} \right|^2 + \left| \frac{\partial C}{\partial t} \right|^2 + \sum_{i,j=1}^3 \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 + \sum_{i,j=1}^3 \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) \right. \\
+ s\varphi(|\nabla z|^2 + |\nabla C|^2) + s^3 \varphi^3 (|z|^2 + |C|^2) \left. dx \, dt \right)
\leq c(\lambda) \left( s^3 \int_{\partial \Omega} e^{2s\alpha} \varphi^3 (|z|^2 + |C|^2) dx \, dt \\
+ s \int_0^T e^{2s\alpha} \varphi^* |g|_{L^2}^2 \, dt + \int_0^T e^{2s\alpha} |g|_{((H^1(\omega))')}^2 \, dt \\
+ \int_0^T e^{2s\alpha} |G|_{((H^1(\omega))')}^2 \, dt \right)
\]

(4.11)

for \( \lambda > \lambda_0, s > s_0(\lambda) \). Inequality (4.11) is the basis on which we shall obtain the desired estimates.
Now let us evaluate the terms in the right-hand side of (4.11) that contain \( w, E, \nabla w, \nabla E, \Delta y_e, \) and \( \Delta B_e \) by means of \( \mu(w, E), M, z, \nabla z, C, \) and \( \nabla C. \)

Since, by the Sobolev imbedding theorem,

\[
|w|_{L^6}^2 + |\nabla w|_{L^6}^2 + |E|_{L^6}^2 + |\nabla E|_{L^6}^2 \leq c\mu^2(w, E),
\]

we have

\[
\begin{align*}
\int_Q e^{2\alpha} |\nabla z|^2 (|w|^2 + |\nabla w|^2) dx \, dt \\
+ \int_Q e^{2\alpha} |\nabla C|^2 (|E|^2 + |\nabla w|^2 + |\nabla E|^2) dx \, dt \\
\leq \int_0^T |e^{\alpha |\nabla z|_{L^3}^2} (|w|_{L^6}^2 + |\nabla w|_{L^6}^2) dt \\
+ \int_0^T |e^{\alpha |\nabla C|^2_{L^3}} (|E|_{L^6}^2 + |\nabla w|_{L^6}^2 + |\nabla E|_{L^6}^2) dt \\
\leq c\mu^2(w, E) \left( \int_0^T |e^{\alpha |\nabla z|_{L^3}^2} dt + \int_0^T |e^{\alpha |\nabla C|^2_{L^3}} dt \right).
\end{align*}
\]

By Hölder’s inequality, we have

\[
|e^{\alpha |\nabla z|_{L^3}^2} | \leq |e^{\alpha |\nabla z|_{L^2}^{1/3} |e^{\alpha |\nabla z|_{L^4}^{2/3}} .
\]

Using Hölder’s inequality once again and the Sobolev imbedding theorem, we obtain

\[
|e^{\alpha |\nabla z|_{L^4}^2} | \leq |e^{\alpha |\nabla z|_{L^6}^{3/4} |e^{\alpha |\nabla z|_{L^2}^{1/4}}
\leq c|e^{\alpha |\nabla z|_{H^1}^{3/4} |e^{\alpha |\nabla z|_{L^2}^{1/4}} .
\]

Combining (4.13) and (4.14), we have

\[
|e^{\alpha |\nabla z|_{L^3}^2} | \leq c|e^{\alpha |\nabla z|_{L^2}^{1/2} |e^{\alpha |\nabla z|_{H^1}^{1/2}} .
\]
We have the same inequality with $C$ replacing $z$. Inserting (4.15) into (4.12), we have

\begin{align*}
\int_Q e^{2s\alpha} |\nabla z|^2 (|w|^2 + |\nabla w|^2) &\, dx \, dt \\
&\quad + \int_Q e^{2s\alpha} |\nabla C|^2 (|E|^2 + |\nabla E|^2) &\, dx \, dt \\
&\leq c_1 \mu^2 (w, E) \int_0^T \left( |e^{s\alpha} \nabla z|_{L^2} |e^{s\alpha} \nabla z|_{H^1} \\
&\quad + |e^{s\alpha} \nabla C|_{L^2} |e^{s\alpha} \nabla C|_{H^1} \right) \, dt \\
&\leq c_2 \mu^2 (w, E) \\
&\times \int_Q e^{2s\alpha} \left(s \varphi^* |\nabla z|^2 + \frac{1}{s \varphi^*} \left( \sum_{i,j=1}^3 \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 + s^2 \lambda^2 \varphi^2 |\nabla z|^2 \right) \\
&\quad + s \varphi^* |\nabla C|^2 \\
&\quad + \frac{1}{s \varphi^*} \left( \sum_{i,j=1}^3 \left( \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 + s^2 \lambda^2 \varphi^2 |\nabla C|^2 \right) \right) \right) \, dx \, dt \\
&\leq c(\lambda) \mu^2 (w, E) \\
&\times \int_Q e^{2s\alpha} \left(s \varphi (|\nabla z|^2 + |\nabla C|^2) + \frac{1}{s \varphi} \sum_{i,j=1}^3 \left( \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \\
&\quad + \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) \right) \, dx \, dt .
\end{align*}

Analogously,

\begin{align*}
\int_Q e^{2s\alpha} (|w|^2 |\nabla C|^2 + |E|^2 |\nabla z|^2) &\, dx \, dt \\
&\leq c(\lambda) \mu^2 (w, E) \int_Q e^{2s\alpha} \left( s \varphi (|\nabla z|^2 + |\nabla C|^2) \\
&\quad + \frac{1}{s \varphi} \sum_{i,j=1}^3 \left( \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 + \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) \right) \, dx \, dt .
\end{align*}
Similarly, we have
\[
\int_Q e^{2\alpha} |z|^2 |\nabla E|^2 \, dx \, dt \leq \int_0^T |e^{\alpha t} z|_{L^2}^2 |\nabla E|_{L^2}^2 \, dx \, dt \\
\leq \mu^2(w, E) \int_Q e^{2\alpha} (|z|^2 + |\nabla z|) \, dx \, dt
\]
(4.18)

and (mind that \( y_e, B_e \in (H^3(\Omega))^3 \))
\[
\int_Q e^{2\alpha} (|\Delta y_e|^2 + |\Delta B_e|^2) (|z|^2 + |C|^2) \, dx \, dt \\
\leq \int_0^T (|\Delta y_e|^2 + |\Delta B_e|^2) (|e^{\alpha t} z|_{L^2}^2 + |e^{\alpha t} C|^2_{L^2}) \, dt \\
\leq c_1 \int_0^T (|e^{\alpha t} z|_{L^2}^2 + |e^{\alpha t} C|^2_{L^2}) \, dt \\
\leq c_2 \int_0^T (|e^{\alpha t} z|^2_{L^2} (|\nabla (e^{\alpha t} z)|^2_{L^2})^{3/2} \\
+ |e^{\alpha t} C|^2_{L^2} (|\nabla (e^{\alpha t} C)|^2_{L^2})^{3/4}) \, dt \\
\leq c_2 \int_0^T (|e^{\alpha t} z|^2_{L^2} + |\nabla (e^{\alpha t} z)|^2_{L^2} + |e^{\alpha t} C|^2_{L^2} + |\nabla (e^{\alpha t} C)|^2_{L^2}) \, dt \\
\leq c(\lambda) \int_Q e^{2\alpha} (|\nabla z|^2 + |\nabla C|^2 + s^2 \varphi^2 (|z|^2 + |C|^2)) \, dx \, dt.
\]
(4.19)

Further, we shall estimate the terms in the right-hand side of (4.11) that contain \( |g|_{L^2}^2, |g|_{(H^1(\omega))^3}^2 \), and \( |G|_{(H^1(\omega))^3}^2 \). To this aim we use the trilinear form \( b \) defined as
\[
b(y^1, y^2, y^3) = \int_\Omega ((y^1 \cdot \nabla) y^2) y^3 \, dx = \sum_{i,j=1}^3 \int_\Omega y_i^1 \frac{\partial y_j^2}{\partial x_i} y_j^3 \, dx.
\]

It is well-known that (see [19, 21]) \( b(y^1, y^2, y^2) = 0 \) for \( y^1 \in V_2 \) and \( y^2 \in (H^1(\Omega))^3 \), whence follows the following symmetry property of \( b \):
\[
b(y^1, y^2, y^3) = -b(y^1, y^3, y^2) \quad \text{for } y^1 \in V_2 \text{ and } y^2, y^3 \in (H^1(\Omega))^3.
\]

Moreover, the following estimate holds (see [6, 19]):
\[
|b(y^1, y^2, y^3)| \leq c |y^1|_{H^{m_1}} |y^2|_{H^{m_2+1}} |y^3|_{H^{m_3}} \quad \text{for } m_1, m_2, m_3 \geq 0
\]
(4.20)

with \( m_1 + m_2 + m_3 > \frac{3}{2} \) or \( m_1 + m_2 + m_3 = \frac{3}{2} \) and at least two \( m_i \) are nonzero, and for \((y^1, y^2, y^3) \in (H^{m_1}(\Omega))^3 \times (H^{m_2+1}(\Omega))^3 \times (H^{m_3}(\Omega))^3\).
It is easy to see that for $\chi \in (L^2(\Omega))^3$,

$$
\int g \cdot \chi \, dx = -b(w + y, z, \chi) + b(E + B, C, \chi) + b(\chi, y, z) + b(\chi, B, C),
$$

$$
\int G \cdot \chi \, dx = -b(w + y, C, \chi) + b(E + B, z, \chi) - b(z, E, \chi) - b(\chi, y, C) - b(\chi, B, z).
$$

Using (4.20), we have for $\gamma > \frac{3}{2}$,

$$
\left| \int g \cdot \chi \, dx \right| \leq c(|w + y_{E|H^\gamma}|z|_{H^1} + |E + B_{E|H^\gamma}|C|_{H^1} + |y_{E|H^2}|z|_{H^1} + |B_{E|H^2}|C|_{H^1})|\chi|_{L^2} \quad \text{a.e. on (0, T) for all } \chi \in (L^2(\Omega))^3,
$$

whence

$$
(4.21) \quad |g|_{L^2} \leq c(\mu(w, E) + 1)(|z|_{H^1} + |C|_{H^1}) \quad \text{a.e. on (0, T)}.
$$

In the same manner we obtain for $\gamma > \frac{3}{2}$

$$
\left| \int g \cdot \chi \, dx \right| \leq c(|w + y_{E|H^\gamma}|z|_{L^2(\omega))^3}|\chi|_{(H^1(\omega))^3} + |E + B_{E|H^\gamma}|C|_{(L^2(\omega))^3}|\chi|_{(H^1(\omega))^3} + |y_{E|H^2}|z|_{(L^2(\omega))^3}|\chi|_{(H^1(\omega))^3} + |B_{E|H^2}|C|_{(L^2(\omega))^3}|\chi|_{(H^1(\omega))^3}) \quad \text{a.e. on (0, T)},
$$

$$
\left| \int G \cdot \chi \, dx \right| \leq c(|w + y_{E|H^\gamma}|z|_{(L^2(\omega))^3}|\chi|_{(H^1(\omega))^3} + |E + B_{E|H^\gamma}|C|_{(L^2(\omega))^3}|\chi|_{(H^1(\omega))^3} + |y_{E|H^2}|z|_{(L^2(\omega))^3}|\chi|_{(H^1(\omega))^3} + |B_{E|H^2}|C|_{(L^2(\omega))^3}|\chi|_{(H^1(\omega))^3}) \quad \text{a.e. on (0, T)}
$$

for all $\chi \in (H^1_0(\omega))^3$, so
\[
|g|_{(H_0^1(\omega))'} \leq c(\mu(w, E) + 1)|z|_{L^2(\omega)} + |C|_{L^2(\omega)} \quad \text{a.e. on } (0, T),
\]
\[
|G|_{(H_0^1(\omega))'} \leq c(\mu(w, E) + 1)|z|_{L^2(\omega)} + |C|_{L^2(\omega)} \quad \text{a.e. on } (0, T).
\]

Now we shall estimate \( s\int_0^T e^{2s\alpha^*\varphi^*} |g|_{L_x^2}^2 \, dt \) by an integral of \(|z|^2 + |C|^2\) (with a certain coefficient) over \(Q\). Set \( \tilde{z} = e^{s\alpha^*} z \) and \( \tilde{C} = e^{s\alpha^*} C \). Then (4.4) can be rewritten as

\[
\frac{\partial \tilde{z}}{\partial t} + \Delta \tilde{z} + ((w + y_e) \cdot \nabla)\tilde{z} - \tilde{z} \cdot (\nabla y_e) = \]

\[
((E + B_e) \cdot \nabla)\tilde{C} + \tilde{C} \cdot (\nabla B_e) + \nabla (e^{s\alpha^*} (\varphi^*)^{1/2})q
\]

\[
+ s\alpha^* (\varphi^*)^{-1} \frac{\partial \varphi^*}{\partial t} \tilde{z} \quad \text{in } Q,
\]

\[
\frac{\partial \tilde{C}}{\partial t} - \text{curl(curl } \tilde{C}) + ((w + y_e) \cdot \nabla)\tilde{C} + \tilde{C} \cdot (\nabla y_e) =
\]

\[
((E + B_e) \cdot \nabla)\tilde{z} - \tilde{z} \cdot (\nabla B_e) - (\tilde{z} \cdot \nabla)E
\]

\[
+ s\alpha^* (\varphi^*)^{-1} \frac{\partial \varphi^*}{\partial t} \tilde{C} \quad \text{in } Q,
\]

\[
\nabla \tilde{z} = 0, \quad \nabla \tilde{C} = 0 \quad \text{in } Q,
\]

\[
\tilde{z} = 0, \quad \tilde{C} \cdot N = 0, \quad (\text{curl } \tilde{C}) \times N = 0 \quad \text{on } \Sigma,
\]

\[
\tilde{z}(\cdot, 0) = \tilde{z}(\cdot, T) = 0, \quad \tilde{C}(\cdot, 0) = \tilde{C}(\cdot, T) = 0 \quad \text{in } \Omega.
\]

Multiplying the first two equations of (4.23) by \( \tilde{z} \) and \( \tilde{C} \), respectively, adding them, and integrating over \(Q\), we obtain

\[
- \int Q |\nabla \tilde{z}|^2 \, dx \, dt - \int Q |\text{curl } \tilde{C}|^2 \, dx \, dt =
\]

\[
\int_0^T (b(\tilde{z}, y_e, \tilde{z}) + b(\tilde{z}, B_e, \tilde{C}) - b(\tilde{C}, y_e, \tilde{C}) - b(\tilde{C}, B_e, \tilde{z}) - b(\tilde{z}, E, \tilde{C}) \, dt
\]

\[
+ s \int_0^T \alpha^* (\varphi^*)^{-1} \frac{\partial \varphi^*}{\partial t} (|\tilde{z}|_{L_x^2}^2 + |\tilde{C}|_{L_x^2}^2) \, dt.
\]

Using the fact that \( \tilde{C} \mapsto (\int Q |\text{curl } \tilde{C}|^2 \, dx)^{1/2} \) is a norm on \( V_2 \) equivalent to the usual norm induced by \((H^1(\Omega))^3\) (see [3] or [19, p. 639]) and estimate (4.20), we
have
\[
\int_Q (|\nabla \tilde{z}|^2 + |\nabla \tilde{C}|^2)\,dx\,dt
\]
\[
\leq c_1 \int_Q (|\tilde{z}|^2 + |\tilde{C}|^2)\,dx\,dt + c(\lambda)s \int_Q (\varphi^*)^{\frac{3}{2}} (|\tilde{z}|^2 + |\tilde{C}|^2)\,dx\,dt
\]
\[
+ c_1 \int_0^T |\tilde{z}|_{H_1} |E|_{H_2} |\tilde{C}|_{H_2} \,dt
\]
\[
\leq c(\lambda) \int_Q (|\tilde{z}|^2 + |\tilde{C}|^2)(1 + s(\varphi^*)^{\frac{3}{2}})\,dx\,dt
\]
\[
+ \frac{1}{2} \int_Q |\nabla \tilde{z}|^2 \,dx\,dt + c_2 M^2 \int_Q |\tilde{C}|^2 \,dx\,dt ,
\]
so
\[
\int_Q (|\nabla \tilde{z}|^2 + |\nabla \tilde{C}|^2)\,dx\,dt \leq c(\lambda) \int_Q (M^2 + 1 + s(\varphi^*)^{\frac{3}{2}})(|\tilde{z}|^2 + |\tilde{C}|^2)\,dx\,dt ,
\]
whence
\[
\int_Q e^{2s\alpha^* \varphi^*} (|\nabla z|^2 + |\nabla C|^2)\,dx\,dt \leq
\]
\[
c(\lambda) \int_Q e^{2s\alpha^* \varphi^*} (M^2 + 1 + s(\varphi^*)^{\frac{3}{2}})(|z|^2 + |C|^2)\,dx\,dt .
\]
Combining the last inequality and (4.21), we obtain for \(s > T^2/2^2\)
\[
s \int_0^T e^{2s\alpha^* \varphi^*} |g|^2_{L^2} \,dt
\]
\[
\leq c(\lambda)(\mu^2(w, E) + 1)
\]
(4.24) \[
\times s \int_Q e^{2s\alpha^* \varphi^*} (M^2 + 1 + s(\varphi^*)^{\frac{3}{2}})(|z|^2 + |C|^2)\,dx\,dt
\]
\[
\leq c(\lambda)(\mu^2(w, E) + 1)(M^2 + 1)s^2 \int_Q e^{2s\alpha^* \varphi^3} (|z|^2 + |C|^2)\,dx\,dt
\]
\[
\leq c(\lambda)(M^2 + 1)(\mu^2(w, E) + 1)s^2 \int_Q e^{2s\alpha \varphi^3} (|z|^2 + |C|^2)\,dx\,dt .
\]
By (4.22), we have
\[
\int_0^T e^{2s\alpha^+\delta} \left(|g|_{((H^1_0(\omega))^3')}^2 + |G|_{((H^1_0(\omega))^3')}^2 \right) dt \leq 
\]
\[
C(\mu^2(w, E) + 1) \int_0^T e^{2s\alpha^+\delta} (|z|^2 + |C|^2) dx dt
\]
(4.25)

Inequalities (4.11), (4.16), (4.19), (4.24), and (4.25) give
\[
\int_0^T e^{2s\alpha} \left( \frac{1}{s\varphi} \left( \left| \frac{\partial z}{\partial t} \right|^2 + \left| \frac{\partial C}{\partial t} \right|^2 \right) + \sum_{i,j=1}^3 \left( \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 + \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) \right) dx dt + s\varphi (|\nabla z|^2 + |\nabla C|^2) + s^3 \varphi^3 (|z|^2 + |C|^2) dx dt
\]
\[
\leq C(\lambda) \left( s^3 \int_{Q_0} e^{2s\alpha \delta} \varphi^3 (|z|^2 + |C|^2) dx dt + \int_{Q} e^{2s\alpha} (|\nabla z|^2 + |\nabla C|^2) dx dt + \mu^2(w, E) \right.
\]
\[
\times \int_{Q} e^{2s\alpha} \left( s\varphi (|\nabla z|^2 + |\nabla C|^2) + \frac{1}{s\varphi} \sum_{i,j=1}^3 \left( \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 + \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) \right) dx dt + (M^2 + 1)(\mu^2(w, E) + 1) s^2 \int_{Q} e^{2s\alpha \delta} \varphi^3 (|z|^2 + |C|^2) dx dt
\]
\[
+ (\mu^2(w, E) + 1) \int_{Q_0} e^{2s\alpha^+\delta} (|z|^2 + |C|^2) dx dt \right)
\]
(4.26)

for \( \lambda > \lambda_0, s > s_0(\lambda) \). We fix \( \lambda = \lambda_1 > \lambda_0 \) and then choose \( s > s_0(\lambda_1) \) such that
\[
(4.27) \quad C(\lambda_1) \int_{Q} e^{2s\alpha} (|\nabla z|^2 + |\nabla C|^2) dx dt \leq \frac{1}{4} s \int_{Q} e^{2s\alpha \varphi} (|\nabla z|^2 + |\nabla C|^2) dx dt,
\]
\[ c(\lambda_1)(M^2 + 1)(\mu^2(w, E) + 1)s^2 \int_Q e^{2s\alpha} \varphi^3(|z|^2 + |C|^2) \, dx \, dt \leq \]
\[ \frac{1}{2} s \int_0^1 e^{2s\alpha} \varphi^3(|z|^2 + |C|^2) \, dx \, dt. \]

(4.28)

Now choosing \( M \) (sufficiently small) such that
\[ c(\lambda_1) M^2 < \frac{1}{4}, \]

(4.29)

we have (mind that \( \frac{1}{2} \leq s \leq 1 \))
\[ c(\lambda_1) \mu^2(w, E) \int_Q e^{2s\alpha} \left( s \varphi(|\nabla z|^2 + |\nabla C|^2) \right. \]
\[ + \frac{1}{s \varphi} \left( \sum_{i,j=1}^3 \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 + \sum_{i,j=1}^3 \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) \right) \, dx \, dt \]
\[ \leq \frac{1}{4} \int_Q e^{2s\alpha} \left( s \varphi(|\nabla z|^2 + |\nabla C|^2) \right. \]
\[ + \frac{1}{s \varphi} \left( \sum_{i,j=1}^3 \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 + \sum_{i,j=1}^3 \left| \frac{\partial^2 C}{\partial x_i \partial x_j} \right|^2 \right) \right) \, dx \, dt. \]

(4.30)

Using (4.27), (4.28), and (4.30) with \( z = z_\varepsilon, C = C_\varepsilon \), inequality (4.26) (with \( z = z_\varepsilon, C = C_\varepsilon \)) yields
\[ \int_Q e^{2s\alpha} \left( \frac{1}{s \varphi} \left| \frac{\partial z_\varepsilon}{\partial t} \right|^2 + \left| \frac{\partial C_\varepsilon}{\partial t} \right|^2 + \sum_{i,j=1}^3 \left( \left| \frac{\partial^2 z_\varepsilon}{\partial x_i \partial x_j} \right|^2 + \left| \frac{\partial^2 C_\varepsilon}{\partial x_i \partial x_j} \right|^2 \right) \right) \]
\[ + s \varphi(|\nabla z_\varepsilon|^2 + |\nabla C_\varepsilon|^2) + s^3 \varphi^3(|z_\varepsilon|^2 + |C_\varepsilon|^2) \right) \, dx \, dt \]
\[ \leq cs^3 \int_{Q_\omega} e^{2s\alpha} \varphi^3(|z_\varepsilon|^2 + |C_\varepsilon|^2) \, dx \, dt \]
\[ + c(M^2 + 1) \int_{Q_\omega} e^{2s\alpha \delta} \varphi^3(|z_\varepsilon|^2 + |C_\varepsilon|^2) \, dx \, dt \quad \text{for } s > s_0(\lambda_1), \]

(4.31)

where \( c = c(\lambda_1) \) (and is independent of \( s \) and \( \varepsilon \)) and \( M \) satisfies (4.29).

Now, to prove the needed boundedness for the controls \( u_\varepsilon \) and \( v_\varepsilon \) and the final states \( y_\varepsilon(T) \) and \( B_\varepsilon(T) \) via (4.4) and (4.5), we have to estimate \( |z_\varepsilon(0)|_{L^2}^2 \) and \( |C_\varepsilon(0)|_{L^2}^2 \) in (4.5) by \( \int_{Q_\omega} e^{2s\alpha \delta} (|z_\varepsilon|^2 + |C_\varepsilon|^2) \, dx \, dt \) (that is, by the first term in the left-hand side of (4.5)). To this end, we multiply the first two equations in (4.4) by
\( z_\varepsilon \) and \( C_\varepsilon \), respectively, and integrate over \( \Omega \); we obtain that
\[
\frac{1}{2} \frac{d}{dt} |z_\varepsilon|^2_{L^2} - |\nabla z_\varepsilon|^2_{L^2} - b(z_\varepsilon, y_\varepsilon, z_\varepsilon) - b(E + B_\varepsilon, C_\varepsilon, z_\varepsilon) - b(z_\varepsilon, B_\varepsilon, C_\varepsilon) = 0 \quad \text{a.e. in } (0, T),
\]
(4.32)
\[
\frac{1}{2} \frac{d}{dt} |C_\varepsilon|^2_{L^2} - |\text{curl } C_\varepsilon|^2_{L^2} - b(E + B_\varepsilon, z_\varepsilon, C_\varepsilon) + b(C_\varepsilon, B_\varepsilon, z_\varepsilon) + b(C_\varepsilon, y_\varepsilon, C_\varepsilon) + b(z_\varepsilon, E, C_\varepsilon) = 0 \quad \text{a.e. in } (0, T).
\]
Adding the two equations in (4.32) and taking the following inequality into account,
\[
|b(z_\varepsilon, y_\varepsilon, z_\varepsilon)| + |b(z_\varepsilon, B_\varepsilon, C_\varepsilon)| + |b(C_\varepsilon, B_\varepsilon, z_\varepsilon)| + |b(C_\varepsilon, y_\varepsilon, C_\varepsilon)| + |b(z_\varepsilon, E, C_\varepsilon)|
\leq c(|z_\varepsilon|^2_{L^2} + |C_\varepsilon|^2_{L^2}) + cM^2|C_\varepsilon|^2_{L^2} + \frac{1}{2} |\nabla z_\varepsilon|^2_{L^2},
\]
we have
\[
\frac{1}{2} \frac{d}{dt} (|z_\varepsilon|^2_{L^2} + |C_\varepsilon|^2_{L^2}) + c(M^2 + 1)(|z_\varepsilon|^2_{L^2} + |C_\varepsilon|^2_{L^2}) \geq 0 \quad \text{a.e. } t \in (0, T),
\]
whence we get
\[
|z_\varepsilon(0)|^2_{L^2} + |C_\varepsilon(0)|^2_{L^2} \leq e^{c(M^2+1)}(|z_\varepsilon(t)|^2_{L^2} + |C_\varepsilon(t)|^2_{L^2}) \quad \text{for } t \in (0, T).
\]
We fix \( t_1 \) and \( t_2 \) such that \( 0 < t_1 < t_2 < T \). There are \( \gamma > 0 \) and \( s_0 > s_0(\lambda_1) \) such that
\[
|z_\varepsilon(x, t)|^2 + |C_\varepsilon(x, t)|^2 \leq \varphi^3 e^{2s\alpha + \gamma s} (|z_\varepsilon(x, t)|^2 + |C_\varepsilon(x, t)|^2)
\]
for \( (x, t) \in \Omega \times (t_1, t_2) \), \( s \geq s_0 \) (that is, \( \varphi^3 e^{2s\alpha + \gamma s} \geq 1 \) for the above \( x, t, \) and \( s \)). Thus, (4.33) gives
\[
e^{\gamma s} \int_{t_1}^{t_2} \int_{\Omega} \varphi^3 e^{2s\alpha} (|z_\varepsilon(x, t)|^2 + |C_\varepsilon(x, t)|^2) dx \, dt
\geq \int_{t_1}^{t_2} \left( |z_\varepsilon(t)|^2_{L^2} + |C_\varepsilon(t)|^2_{L^2} \right) dt
\geq e^{-c(M^2+1)}(t_2 - t_1)(|z_\varepsilon(0)|^2_{L^2} + |C_\varepsilon(0)|^2_{L^2}).
\]
Using (4.31) one obtains for an arbitrary but fixed positive \( \delta' \) that is smaller than \( \delta \) (mind that \( \alpha^* \leq \alpha \)),
\[
|z_\varepsilon(0)|^2_{L^2} + |C_\varepsilon(0)|^2_{L^2} \leq c(s)(M^2 + 1)e^{c(M^2+1)} \int_{Q_w} e^{2s\alpha\delta'} (|z_\varepsilon|^2 + |C_\varepsilon|^2) dx \, dt
\]
for $s \geq s_0$. This estimate and (4.5) with $\delta = \delta'$ yield

\begin{equation}
\frac{1}{2} \int_{Q_0} e^{2s\alpha \delta'} \left( |z_{\varepsilon}|^2 + |C_{\varepsilon}|^2 \right) dx \, dt + \frac{1}{\varepsilon} \left( |y_{\varepsilon}(T)|_{L^2}^2 + |B_{\varepsilon}(T)|_{L^2}^2 \right) \leq 2c(s)(M^2 + 1)e^{c(M^2+1)}(|y_{\varepsilon}|_{L^2}^2 + |B_{\varepsilon}|_{L^2}^2)
\end{equation}

for $s \geq s_0$.

Hence, using (4.3) (we take $\delta = \delta'$ in $(P_{\varepsilon})$), we have

\begin{equation}
\frac{1}{2} \int_{Q} \left( |u_{\varepsilon}|^2 + |v_{\varepsilon}|^2 \right) dx \, dt + \frac{1}{\varepsilon} \left( |y_{\varepsilon}(T)|_{L^2}^2 + |B_{\varepsilon}(T)|_{L^2}^2 \right) \leq 2c(s)(M^2 + 1)e^{c(M^2+1)}(|y_{\varepsilon}|_{L^2}^2 + |B_{\varepsilon}|_{L^2}^2)
\end{equation}

for $s \geq s_0$.

We also have

\begin{equation}
\int_{Q} \left( \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 + \left| \frac{\partial v_{\varepsilon}}{\partial t} \right|^2 \right) dx \, dt
\end{equation}

\begin{equation}
\leq c_1 \int_{Q} e^{4s\alpha \delta} \left( \left| \frac{\partial z_{\varepsilon}}{\partial t} \right|^2 + \left| \frac{\partial C_{\varepsilon}}{\partial t} \right|^2 \right) + s^2 \varphi^3 \left( |z_{\varepsilon}|^2 + |C_{\varepsilon}|^2 \right) dx \, dt
\end{equation}

\begin{equation}
\leq c_2 \int_{Q} e^{2s\alpha} \left( |z_{\varepsilon}|^2 + |C_{\varepsilon}|^2 + \frac{1}{s\varphi} \left( \left| \frac{\partial z_{\varepsilon}}{\partial t} \right|^2 + \left| \frac{\partial C_{\varepsilon}}{\partial t} \right|^2 \right) \right) dx \, dt.
\end{equation}

Using (4.31) and (4.34) once again, we obtain that

\begin{equation}
\int_{Q} \left( \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 + \left| \frac{\partial v_{\varepsilon}}{\partial t} \right|^2 \right) dx \, dt
\end{equation}

\begin{equation}
\leq c(s)(M^2 + 1) \int_{Q_0} e^{2s\alpha \delta'} \left( |z_{\varepsilon}|^2 + |C_{\varepsilon}|^2 \right) dx \, dt
\end{equation}

\begin{equation}
\leq 4c(s)(M^2 + 1)^2 e^{c(M^2+1)}(|y_{\varepsilon}|_{L^2}^2 + |B_{\varepsilon}|_{L^2}^2)
\end{equation}

for $s \geq s_0$.

By (4.35) and (4.36), we have

\begin{equation}
|u_{\varepsilon}|_{H^1(0,T;L^2(\Omega))^3}^2 + |v_{\varepsilon}|_{H^1(0,T;L^2(\Omega))^3}^2 + \frac{1}{\varepsilon} \left( |y_{\varepsilon}(T)|_{L^2}^2 + |B_{\varepsilon}(T)|_{L^2}^2 \right) \leq \beta(M^2 + 1)^2 e^{c(M^2+1)}(|y_{\varepsilon}|_{L^2}^2 + |B_{\varepsilon}|_{L^2}^2),
\end{equation}

where $\beta = 4c(s_0)$ (which is independent of $\varepsilon$) and $M$ satisfies (4.29).

Now we shall prove that the sequence $\{(y_{\varepsilon}, B_{\varepsilon})\}$ is bounded in

\[(L^2(0,T; D(A_1)) \cap H^1(0,T; V_1)) \times (L^2(0,T; D(A_2)) \cap H^1(0,T; V_2)).\]
First we rewrite (2.5) as
\[ y' + A_1y + P\left( ((w + y_e) \cdot \nabla) y \right) + P\left( ((y \cdot \nabla)y_e) + P(E \cdot (\nabla B)) \right) \\
- P\left( ((E + B_e) \cdot \nabla) B \right) - P\left( (B \cdot \nabla) B_e \right) = P(mu) \quad \text{in} \ (0, T), \]
\[ (4.38) \]
\[ B' + A_2B + P\left( ((w + y_e) \cdot \nabla) B \right) + P((y \cdot \nabla) B_e) \\
- P\left( ((E + B_e) \cdot \nabla) y \right) - P((B \cdot \nabla) y_e) = P(mv) \quad \text{in} \ (0, T). \]
\[ (4.39) \]
Multiplying (4.38) scalarly in \( H \) by \( y \) and \( B \), respectively, integrating over \( (0, t) \), and adding the two obtained equations, we have
\[ \frac{1}{2}|y(t)|^2_{L^2} + \frac{1}{2}|B(t)|^2_{L^2} + \int_0^t |\nabla y|^2_{L^2} \, d\tau + \int_0^t |\text{curl } B|^2_{L^2} \, d\tau \\
+ \int_0^t \left( b(y, y_e, y) + b(y, B, E) - b(B, B_e, y) \\
+ b(y, B_e, B) - b(B, y_e, B) \right) \, d\tau \\
= \frac{1}{2}|y|^2_{H^1} + \frac{1}{2}|B|^2_{H^1} + \int_0^t \int_\Omega (mu \cdot y + mv \cdot B) \, dx \, d\tau. \]
\[ (4.40) \]
Analogously, multiplying (4.38) scalarly in \( H \) by \( A_1y \) and \( A_2B \), respectively, we obtain
\[ \frac{1}{2}|y(t)|^2_{H^1} + \frac{1}{2}|\text{curl } B(t)|^2_{L^2} + \int_0^t \left( |A_1y|^2_{L^2} + |A_2B|^2_{L^2} \right) \, d\tau \\
+ \int_0^t \left( b(w + y_e, y, A_1y) + b(y, y_e, A_1y) + b(A_1y, B, E) \\
- b(E + B_e, B, A_1y) - b(B, B_e, A_1y) + b(w + y_e, B, A_2B) \\
- b(E + B_e, y, A_2B) + b(y, B_e, A_2B) - b(B, y_e, A_2B) \right) \, d\tau \\
= \frac{1}{2}|y|^2_{H^1} + \frac{1}{2}|\text{curl } B|^2_{H^1} + \int_0^t \int_\Omega (mu \cdot A_1y + mv \cdot A_2B) \, dx \, d\tau. \]
\[ (4.41) \]
Using (4.20) with different \( m_1, m_2, \) and \( m_3 \) and the fact that \((w, E) \in K\), from (4.39) and (4.40), we deduce (do not forget that \( B \mapsto |\text{curl } B|_{(L^2(\Omega))^3} \) is equivalent to the usual norm of \((H^1(\Omega))^3\))
\[ |y(t)|^2_{L^2} + |B(t)|^2_{L^2} + \int_0^t \left( |y|^2_{H^1} + |B|^2_{H^1} \right) \, d\tau \\
\leq c\left( |y|^2_{L^2} + |B|^2_{L^2} \right) + c \int_0^t \left( |u|^2_{L^2} + |v|^2_{L^2} \right) \, d\tau \\
+ c(M^2 + 1) \int_0^t \left( |y|^2_{L^2} + |B|^2_{L^2} \right) \, d\tau, \]
\begin{align}
|y(t)|_{H^1}^2 + |B(t)|_{H^1}^2 + \int_0^t (|A_1 y|_{L^2}^2 + |A_2 B|_{L^2}^2) d\tau \\
\leq c(|y^0|_{H^1}^2 + |B^0|_{H^1}^2) + c \int_0^t (|u|_{L^2}^2 + |v|_{L^2}^2) d\tau \\
+ c \int_0^t (|y|_{L^2}^2 + |B|_{L^2}^2) d\tau + c(M^2 + 1) \int_0^t (|y|_{H^1}^2 + |B|_{H^1}^2) d\tau.
\end{align}

By Gronwall’s inequality, (4.41) yields

\begin{align}
|y(t)|_{L^2}^2 + |B(t)|_{L^2}^2 + \int_0^t (|y|_{H^1}^2 + |B|_{H^1}^2) d\tau \leq \\
c(M^2 + 1)e^{c(M^2+1)}(|y^0|_{L^2}^2 + |B^0|_{L^2}^2 + \int_0^t (|u|_{L^2}^2 + |v|_{L^2}^2) d\tau)
\end{align}

for \( t \in [0, T] \). Inequalities (4.42) and (4.43) give

\begin{align}
|y(t)|_{H^1}^2 + |B(t)|_{H^1}^2 + \int_0^t (|A_1 y|_{L^2}^2 + |A_2 B|_{L^2}^2) d\tau \\
\leq c(|y^0|_{H^1}^2 + |B^0|_{H^1}^2) \\
+ c(M^2 + 1)^2 e^{c(M^2+1)}(|y^0|_{L^2}^2 + |B^0|_{L^2}^2 + \int_0^t (|u|_{L^2}^2 + |v|_{L^2}^2) d\tau)
\end{align}

for \( t \in [0, T] \). Taking (4.37) into account, we have, for \( t \in [0, T] \),

\begin{align}
|y_x(t)|_{H^1}^2 + |B_x(t)|_{H^1}^2 + \int_0^T (|A_1 y_x|_{L^2}^2 + |A_2 B_x|_{L^2}^2) d\tau \leq \\
c(|y^0|_{H^1}^2 + |B^0|_{H^1}^2) + c\beta (M^2 + 1)^4 e^{c(M^2+1)}(|y^0|_{L^2}^2 + |B^0|_{L^2}^2)
\end{align}

Multiplying (4.38) scalarly in \( H \) by \( y' \) and \( B' \), respectively, we obtain after some calculations involving (4.20) (for some \( \gamma > \frac{3}{2} \))

\begin{align}
\int_0^T |y'|_{L^2}^2 d\tau + \int_0^T |B'|_{L^2}^2 d\tau \\
\leq c \left( \int_0^T |A_1 y|_{L^2}^2 d\tau + \int_0^T |w + y^2 y'|_{H^1} y_{H^1} d\tau + \int_0^T |y|_{L^2}^2 d\tau \\
+ \int_0^T |E|_{H^1} |B|_{H^1}^2 d\tau + \int_0^T |E + B e|_{H^1} |B|_{H^1}^2 d\tau + \int_0^T |B|_{L^2}^2 d\tau \\
+ \int_0^T |u|_{L^2}^2 d\tau + \int_0^T |A_2 B|_{L^2}^2 d\tau + \int_0^T |w + y^2 y'|_{H^1} |B|_{H^1}^2 d\tau \\
+ \int_0^T |E + B e|_{H^1} |y|_{H^1}^2 d\tau + \int_0^T |v|_{L^2}^2 \right).
\end{align}
By (4.43), (4.45), and (4.37), the last inequality gives

\[
\int_0^T (|y_\varepsilon'|_L^2 + |B'_\varepsilon'|_L^2) \, dt \leq c \left( |y_0|_{H^1}^2 + |B_0|_{H^1}^2 + (M^2 + 1)^5 e^{(M^2+1)} \left( |y_0|_{L^2}^2 + |B_0|_{L^2}^2 \right) \right).
\]

Multiplying the two equations in (4.38) scalarly in \( H \) by \( A_1 y' \) and \( A_2 B' \), respectively, and integrating over \( (0, t) \), one obtains after some calculation

\[
\int_0^t \left| \nabla y' \right|^2 L^2 \, d\tau + \frac{1}{2} |A_1 y(t)|^2_{L^2} - \frac{1}{2} |A_1 y^0|^2_{L^2}
= - \int_0^t b(w + y_e, y, A_1 y') \, d\tau - \int_0^t b(y, y_e, A_1 y') \, d\tau
- \int_0^t b(A_1 y', B, E) \, d\tau + \int_0^t b(E + B_e, B, A_1 y') \, d\tau
+ \int_0^t b(B, B_e, A_1 y') \, d\tau + \int_0^t \int_{\Omega} \mu_1 \cdot A_1 y' \, dx \, d\tau,
\]

(4.47)

\[
\int_0^t |\text{curl } B'|^2 L^2 \, d\tau + \frac{1}{2} |A_2 B(t)|^2_{L^2} - \frac{1}{2} |A_2 B_0|^2_{L^2}
= - \int_0^t b(w + y_e, B, A_2 B') \, d\tau - \int_0^t b(y, B_e, A_2 B') \, d\tau
+ \int_0^t b(E + B_e, y, A_2 B') \, d\tau + \int_0^t b(B, y_e, A_2 B') \, d\tau
+ \int_0^t \int_{\Omega} \mu_2 \cdot A_2 B' \, dx \, d\tau.
\]

Let us estimate each term from the right-hand sides of equations (4.47). Letting \( y > \frac{3}{2} \), integrating by parts, and using estimate (4.20), the interpolation inequality

\[ |y|_{H^{3/2}} \leq c |y|_{H^1}^{1/2} |y|_{H^2}^{1/2} \quad \text{for } y \in (H^2(\Omega))^3, \]

the first of the following two (well-known) estimates (see (2.4) and (2.8) in [19])

(4.48)

\[ |y|_{(H^2(\Omega))^3} \leq c |A_1 y|_{(L^2(\Omega))^3} \quad \text{for } y \in D(A_1), \]
\[ |B|_{(H^2(\Omega))^3} \leq c |A_2 B|_{(L^2(\Omega))^3} \quad \text{for } B \in D(A_2), \]
and inequality (4.45), we obtain

\[
\left| \int_0^t b(w + y_e, y, A_1y')d\tau \right|
\leq \int_0^t \left( |b(w', y, A_1y)| + |b(w + y_e, y', A_1y)| \right) d\tau
\]
\[
+ |b(w(0) + y_e, y^0, A_1y^0)| + |b(w(t) + y_e, y(t), A_1y(t))|
\leq c_1 \int_0^t \left( |w'|_{H^1} |y|_{H^{3/2}} + |y'|_{H^1} |w + y_e|_{H^r} \right) |A_1y|_{L^2} d\tau
\]
\[
+ c_1 \left( |w(0) + y_e|_{H^r} |y^0|_{H^1} |A_1y^0|_{L^2}
\right)
\leq c_2 \left( \int_0^t |w'|_{H^1} |y|_{H^1} d\tau \right)^{1/2} \left( \int_0^t |A_1y|_{L^2}^2 d\tau \right)^{1/2}
\]
\[
+ c_2(M + 1) \int_0^t |y'|_{H^1} |A_1y|_{L^2} d\tau
\]
\[
+ \frac{1}{12c_0} |A_1y(t)|_{L^2}^2 + c_2(M^2 + 1) |y(t)|_{H^1}^2
\]
\[
+ c_2(M + 1) \left( |y^0|_{H^1}^2 + |A_1y^0|_{L^2}^2 \right)
\leq \frac{1}{12c_0} \left( |A_1y(t)|_{L^2}^2 + \int_0^t |y'|_{H^1}^2 d\tau \right) + c_3 \int_0^t |A_1y|_{L^2}^3 d\tau
\]
\[
+ \left( |y^0|_{H^2}^2 + |B^0|_{H^2}^2 \right) \left( c_1(M) + c_2(M) (|y^0|_{H^2} + |B^0|_{H^2}) \right),
\]

where \( c_0 \) is a constant such that

\[
|y|_{H^1}^2 \leq c_0 |\nabla y|_{L^2}^2 \quad \text{for} \; y \in V_1, \quad |B|_{H^1}^2 \leq c_0 |\text{curl} \; B|_{L^2}^2 \quad \text{for} \; y \in V_2.
\]

Analogously, we have

\[
\left| \int_0^t b(E + B_e, y, A_2B')d\tau \right|
\leq \frac{1}{12c_0} \left( |A_2B(t)|_{L^2}^2 + \int_0^t |y'|_{H^1}^2 d\tau \right) + c_3 \int_0^t |A_1y|_{L^2}^2 |A_2B|_{L^2} d\tau
\]
\[
+ \left( |y^0|_{H^2}^2 + |B^0|_{H^2}^2 \right) \left( c_1(M) + c_2(M) (|y^0|_{H^2} + |B^0|_{H^2}) \right).
\]
\[
\| \int_0^t b(w + y_e, B, A_2 B')d\tau \| \\
\leq \frac{1}{12c_0} \left( |A_2 B(t)|_{L^2}^2 + \int_0^t |B'|_{H^1}^2 d\tau \right) + c \int_0^t |A_2 B|^3_{L^2} d\tau \\
+ \left( |y^0|_{H^2} + |B^0|_{H^2} \right) \left( c_1(M) + c_2(M) (|y^0|_{H^2} + |B^0|_{H^2}) \right),
\]

(4.51)

\[
\| \int_0^t b(E + B_e, B, A_1 y')d\tau \| \\
\leq \frac{1}{12c_0} \left( |A_1 y(t)|_{L^2}^2 + \int_0^t |B'|_{H^1}^2 d\tau \right) + c \int_0^t |A_2 B|^2_{L^2} |A_1 y|_{L^2} d\tau \\
+ \left( |y^0|_{H^2} + |B^0|_{H^2} \right) \left( c_1(M) + c_2(M) (|y^0|_{H^2} + |B^0|_{H^2}) \right),
\]

(4.52)

\[
\| \int_0^t b(A_1 y', B, E)d\tau \| \\
\leq \frac{1}{12c_0} \left( |A_1 y(t)|_{L^2}^2 + \int_0^t |B'|_{H^1}^2 d\tau \right) \\
+ \left( |y^0|_{H^2} + |B^0|_{H^2} \right) \left( c_1(M) + c_2(M) (|y^0|_{H^2} + |B^0|_{H^2}) \right).
\]

(4.53)

In the same way as before, but also using the fact that \( \nabla y_e \in (L^\infty(\Omega))^9 \) (mind that \( y_e \in (H^3(\Omega))^9 \)) and (4.46), we obtain

\[
\| \int_0^t b(y, y_e, A_1 y')d\tau \| \\
\leq c \left( \int_0^t |y'|_{L^2} |A_1 y|_{L^2} d\tau + |A_1 y(t)|_{L^2} |y(t)|_{L^2} + |y^0|_{L^2} |A_1 y^0|_{L^2} \right) \\
\leq \frac{1}{12c_0} \left( |A_1 y(t)|_{L^2}^2 + \int_0^t |y'|_{H^1}^2 d\tau \right) + c(M) (|y^0|_{H^2}^2 + |B^0|_{H^2}^2).
\]

(4.54)

Similarly, we have

\[
\| \int_0^t b(B, B_e, A_1 y')d\tau \| \leq \\
\frac{1}{12c_0} \left( |A_1 y(t)|_{L^2}^2 + \int_0^t |B'|_{H^1}^2 d\tau \right) + c(M) (|y^0|_{H^2}^2 + |B^0|_{H^2}^2),
\]

(4.55)

\[
\| \int_0^t b(y, B_e, A_2 B')d\tau \| \leq \\
\frac{1}{12c_0} \left( |A_2 B(t)|_{L^2}^2 + \int_0^t |y'|_{H^1}^2 d\tau \right) + c(M) (|y^0|_{H^1}^2 + |B^0|_{H^1}^2),
\]

(4.56)
As above, by (4.37) and (4.45), we obtain

\[ \left| \int_0^t b(B, y_e, A_2 B') d\tau \right| \leq \frac{1}{12c_0} \left( |A_2 B(t)|_{L^2}^2 + \int_0^t |B'|_{H^1}^2 \, d\tau \right) + c(M) \left( |y_0^0|_{H^2}^2 + |B_0^0|_{H^2}^2 \right). \]

As above, by (4.37) and (4.45), we obtain

\[ \left| \int_0^t \int \mu \cdot A_1 y' \, dx \, d\tau \right| \]

\[ \leq \int_0^t \int \left| \mu' \cdot A_1 y \right| \, dx \, d\tau \]

\[ + \int \Omega |\mu(t) \cdot A_1 y(t)| \, dx + \int \Omega |\mu(0) \cdot A_1 y_0^0| \, dx \]

\[ \leq \frac{1}{2} \int_0^t |\mu'|_{L^2}^2 \, d\tau + \frac{1}{2} \int_0^t |A_1 y|_{L^2}^2 \, d\tau \]

\[ + c|u(t)|_{L^2}^2 + \frac{1}{12c_0} |A_1 y(t)|_{L^2}^2 + \frac{1}{2} |u(0)|_{L^2}^2 + \frac{1}{2} |A_1 y_0^0|_{L^2}^2 \]

\[ \leq \frac{1}{12c_0} |A_1 y(t)|_{L^2}^2 + c(M) \left( |y_0^0|_{H^2}^2 + |B_0^0|_{H^2}^2 \right). \]

Similarly, we have

\[ \left| \int_0^t \int \nu \cdot A_2 B' \, dx \, d\tau \right| \leq \frac{1}{12c_0} |A_2 B(t)|_{L^2}^2 + c(M) \left( |y_0^0|_{H^2}^2 + |B_0^0|_{H^2}^2 \right). \]

Now, using (4.49) through (4.59) in equations (4.47), we obtain

\[ \int_0^t \left( |y'|_{H^1}^2 + |B'|_{H^1}^2 \right) \, d\tau + |A_1 y(t)|_{L^2}^2 + |A_2 B(t)|_{L^2}^2 \]

\[ \leq \left( |y_0^0|_{H^2} + |B_0^0|_{H^2} \right) \left( c_1(M) + c_2(M) \left( |y_0^0|_{H^2} + |B_0^0|_{H^2} \right) \right) \]

\[ + c \int_0^t \left( |A_1 y|_{L^2}^2 + |A_2 B|_{L^2}^2 \right) \left( |A_1 y'|_{L^2}^2 + |A_2 B'|_{L^2}^2 \right) \, d\tau. \]

Applying the Gronwall inequality to (4.60) and using (4.45), we have

\[ |A_1 y_e(t)|_{L^2}^2 + |A_2 B_e(t)|_{L^2}^2 \]

\[ \leq \left( |y_0^0|_{H^2} + |B_0^0|_{H^2} \right) \left( c_1(M) + c_2(M) \left( |y_0^0|_{H^2} + |B_0^0|_{H^2} \right) \right) \]

\[ \times \exp \int_0^t \left( |A_1 y_e|_{L^2}^2 + |A_2 B_e|_{L^2}^2 \right) \, d\tau \]

\[ \leq \left( |y_0^0|_{H^2} + |B_0^0|_{H^2} \right) \Phi(M, |y_0^0|_{H^2} + |B_0^0|_{H^2}) \quad \text{for} \quad t \in [0, T], \]
where $\Phi(\cdot, \cdot)$ is a positive function that is increasing with respect to the second variable. Coupling (4.60) and (4.61), we obtain

\begin{equation}
(4.62) \quad \int_0^T \left( |y_\varepsilon'|^2_{H^1} + |B_\varepsilon'|^2_{H^1} \right) dt \leq \left( |y^0|_{H^2} + |B^0|_{H^2} \right) \Phi \left( M, |y^0|_{H^2} + |B^0|_{H^2} \right).
\end{equation}

Now we view equations (2.5) with $y = y_\varepsilon$, $B = B_\varepsilon$, $p = p_\varepsilon$, $u = u_\varepsilon$, and $v = v_\varepsilon$ as an elliptic system ($\partial y/\partial t$ and the first-order terms go in the right-hand side). By a well-known regularity result for the steady state Stokes equations with null boundary conditions (see [21, propositions 2.2 and 2.3] or [6, theorem 3.11]), we have $p_\varepsilon(t) \in H^1(\Omega)$ and

\begin{equation}
|p_\varepsilon(t)|^2_{H^1(\Omega)} \leq c \left( (M^2 + 1) \left( |y_\varepsilon(t)|^2_{H^1} + |B_\varepsilon(t)|^2_{H^1} \right) + |u_\varepsilon(t)|^2_{L^2} + \left| \frac{\partial y_\varepsilon}{\partial t} (\cdot, t) \right|^2_{L^2} \right)
\end{equation}
a.e. $t \in (0, T)$. Therefore, by (4.37), (4.45), and (4.46),

\begin{equation}
\inf_{\rho \in \mathbb{R}} \int_\Omega |p_\varepsilon + \rho|^2 dx dt + \int_\Omega |\nabla p_\varepsilon|^2 dx dt \leq \int_\Omega c \left( (M^2 + 1) \left( |y^0|^2_{H^1} + |B^0|^2_{H^1} \right) + (M^2 + 1)^5 e^{c(M^2+1)} \left( |y^0|^2_{L^2} + |B^0|^2_{L^2} \right) \right).
\end{equation}

Inequalities (4.45), (4.48), and (4.62) show that $\{(y_\varepsilon, B_\varepsilon)\}$ is bounded in

\begin{equation}
(L^\infty(0, T; D(A_1)) \cap H^1(0, T; V_1)) \times (L^\infty(0, T; D(A_2)) \cap H^1(0, T; V_2)).
\end{equation}

Besides, the inclusion $(L^2(0, T; D(A_1)) \times L^2(0, T; D(A_2))) \cap (H^1(0, T; H))^2 \subset L^2(0, T; V_1) \times L^2(0, T; V_2)$ is compact. Consequently, on a subsequence, we have

\begin{equation}
\left( \frac{\partial y_\varepsilon}{\partial t}, \frac{\partial B_\varepsilon}{\partial t} \right) \to \left( \frac{\partial y}{\partial t}, \frac{\partial B}{\partial t} \right)
\end{equation}
weakly in $L^2(0, T; V_1 \times V_2)$

\begin{equation}
\text{curl } B_\varepsilon \to \text{curl } B
\end{equation}
weakly in $L^2(\Sigma)$

for some

\begin{equation}
(y, B) \in \left( L^\infty(0, T; D(A_1)) \cap H^1(0, T; V_1) \right) \times \left( L^\infty(0, T; D(A_2)) \cap H^1(0, T; V_2) \right).
\end{equation}
From (4.37) and (4.63) it follows that
\[(u_\varepsilon, v_\varepsilon) \to (u, v) \text{ weakly in } H^1(0, T; (L^2(\Omega))^6),\]
\[\nabla p_\varepsilon \to \nabla p \text{ weakly in } L^2(Q),\]
for some \((u, v) \in H^1(0, T; (L^2(\Omega))^6)\) and \(p \in L^2(0, T; H^1(\Omega))\). Now letting \(\varepsilon\) tend to zero in (2.5) with \(y = y_\varepsilon, B = B_\varepsilon, p = p_\varepsilon, u = u_\varepsilon,\) and \(v = v_\varepsilon\), we derive that \(y, B, p, u,\) and \(v\) satisfy (2.5); again using (4.37), it follows that \(y(T) = 0, B(T) = 0\) a.e. in \(\Omega\) (that is, (4.1) is fulfilled), and \(u\) and \(v\) satisfy (4.2) for some \(\beta > 0\) independent of \((w, E) \in K\). This finishes the proof of Lemma 4.1.

Now we are prepared to prove the local exact null controllability for system (2.4) by using Kakutani’s fixed-point theorem. We define the map \(\Gamma : K \to (L^2(Q))^3 \times (L^2(Q))^3\) by
\[
\Gamma(w, E) = \{ (y, B) \in (L^\infty(0, T; D(A_1)) \cap H^1(0, T; V_1)) \times (L^\infty(0, T; D(A_2)) \cap H^1(0, T; V_2)) : (y, B) \text{ are the solutions of (2.5) corresponding to } (w, E) \text{ and to some } (u, v) \in H^1(0, T; (L^2(\Omega))^6) \text{ satisfying (4.2) in Lemma 4.1 such that } y(T) = 0 \text{ and } B(T) = 0 \text{ a.e. in } \Omega \}\.
\]

First we shall prove that \(\Gamma(K) \subset K\) for \(|y_0^0|_{H^2} + |B_0^0|_{H^2}\) sufficiently small. By (4.61), (4.62), and (4.48) we have
\[
|y|^2_{H^1(0, T; V_1)} + |B|^2_{H^1(0, T; V_2)} + |y|_{L^\infty(0, T; (H^2(\Omega))^3)} + |B|_{L^\infty(0, T; (H^2(\Omega))^3)} \leq \left( |y_0^0|_{H^2} + |B_0^0|_{H^2} \right) \Phi(M, |y_0^0|_{H^2} + |B_0^0|_{H^2}).
\]
Therefore there exists \(\eta > 0\) sufficiently small such that for \(|y_0^0|_{H^2} + |B_0^0|_{H^2} < \eta\) we have \(\mu(y, B) \leq M\). So, we have shown that \(\Gamma(K) \subset K\).

According to Lemma 4.1, \(\Gamma(w, E)\) is nonempty for each \((w, E) \in K\), and by (4.45), (4.46), and (4.63) it follows as before that \(\Gamma\) is a closed-set-valued map. More specifically, if \(K \ni (w_n, E_n) \to (w, E)\) strongly in \((L^2(Q))^6\) and, for some \((u_n, v_n)\) satisfying (4.2), \(\Gamma(w_n, E_n) \ni (y_n, B_n) \to (y, B)\) strongly in \((L^2(Q))^6\), then \((y, B) \in \Gamma(w, E)\); i.e., there exists \((u, v)\) satisfying (4.2) such that the corresponding solution \((y, B)\) belongs to \(\Gamma(w, E)\). (The latter follows by a standard device from the estimates obtained above on the solution \((y, B)\) of equations (4.38).) Since the range of the closed-set-valued map \(\Gamma\) is compact in \((L^2(Q))^6\), it follows that \(\Gamma\) is upper-semicontinuous.

By Kakutani’s fixed-point theorem there exists \((y^*, B^*) \in K\) such that \((y^*, B^*) \in \Gamma(y^*, B^*)\). Consequently, \((y^*, B^*)\) and the corresponding controller \((u^*, v^*)\) solve the null controllability problem for system (2.4). Thus Theorem 2.1 is completely proven.
Bibliography


Received March 2002.