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Theory of Harmonic Grid Generation

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Harmonic grid generation methods for multiply connected plane regions and regions on curved surfaces are discussed. These methods are analysed by unifying them to a general mapping problem on a Riemann surface. In particular, using this formulation it is proven that these mappings are globally one-to-one and on to.

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1. INTRODUCTION

One of the most important tasks of computational physics is the generation of boundary conforming grids in the computational domain. This problem can be framed mathematically as the construction of certain coordinate charts on a given Riemannian manifold. In this paper the method of harmonic mappings for conformally flat manifolds is considered. Harmonic grid generation methods are extensively used in computational physics. The basic idea of generating grids using harmonic mappings for subdomains of Euclidean spaces was introduced by Winslow [14] and developed by others [1], [3], [11]. Harmonic mapping has also been used to generate grids on portions of curved surfaces by one of the authors [9], [10]. Warsi [12], [13] has shown that such mapping methods can be used to generate grid surfaces for three dimensional domains. Although computational work done in this

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subject is extensive the underlying mathematical problem is yet to be studied. The paper by Godunov and Prokopov [3] was the first one to address a mathematical question with regard to the validity of the harmonic mapping method. They showed that harmonic mapping on the plane is locally one-to-one. Mastin and Thompson [5] provided an alternative and slightly more complete proof of the same theorem.

In this paper we extend these earlier works in a number of aspects. We prove that these mappings are globally one-to-one by providing a lemma that is valid for higher dimensions and for nonharmonic mappings. We also provide a slightly different proof of the local homeomorphism and indicate the smoothness requirement of the boundary data that appears to be necessary for these results. The results are generalized by unifying the harmonic mapping methods on multiply connected domains and on portions of analytic curved surfaces using the elegant uniformization theorem. The restriction on the shape of the mapped domain is relaxed to be only of convex type.

2. HARMONIC MAPPING METHOD FOR AN ANALYTIC RIEMANNIAN MANIFOLD

The simplest grid generation method of this kind can be formulated as follows: Let Ω be a simply connected region on an n-dimensional Riemannian manifold M^n with compact closure (see Figure 1). The problem is to find a boundary conforming coordinate system x^i, $i = 1, \ldots, n$. This is accomplished by first mapping $\partial\Omega_0$ in a one-to-one manner to the boundary of Ω_1 in the x^i-space. To map the interior of Ω_0 we use the following conditions

$$\Delta x^i = 0, \qquad i = 1, 2, \ldots, n,$$

where Δ is the Laplace–Beltrami operator. The harmonic condition ensures the existence and uniqueness of these coordinate functions. The smoothness of x^i depends on the smoothness of the boundary data. For example, if the boundary values of x^i are continuously differentiable then we have by the classical potential theory and Weyl's lemma [6], $x^i \in c^1(\bar{\Omega}) \cap C^{\prime}(\Omega)$.

2.2. Formulation on a Given Conformally Flat Manifold

The derivation of the grid generation equation is based on the following:

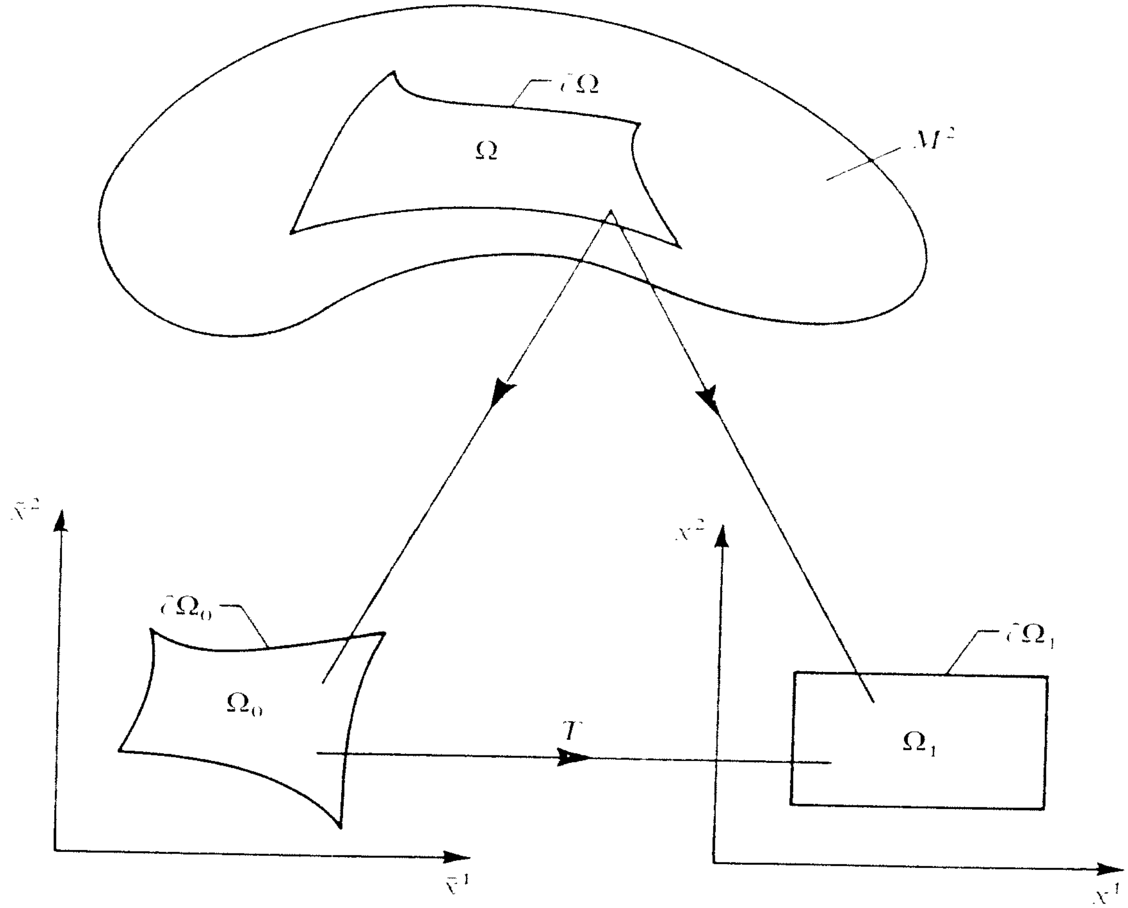

FIGURE 1.

THEOREM 1 *Let U_0 be a coordinate patch on a conformally flat Riemannian manifold M^n with isothermal coordinate system $\bar{X}^\alpha$, $\alpha = 1, \ldots, n$ and metric tensor $\{\lambda(\bar{X}^1, \bar{X}^2, \ldots, \bar{X}^n)\}^2 \delta_{\alpha\beta}$, $\alpha = 1, \ldots, n$, $\beta = 1, \ldots, n$. Suppose U_1 is an overlapping coordinate patch with coordinate system x^α, $\alpha = 1, \ldots, n$ and metric tensor $g^{\alpha\beta}$, $\alpha = 1, \ldots, n$, $\beta = 1, \ldots, n$, such that $\Delta x^\alpha = 0$ for $\alpha = 1, \ldots, n$, where Δ is the Laplace–Beltrami operator. Then the isothermal coordinates satisfy*

$$g^{\alpha\beta} \frac{\partial^2 \bar{X}^\gamma}{\partial x^\alpha \, \partial x^\beta} - \frac{(n-2)}{2\lambda^4} \delta^{\theta\gamma} \frac{\partial \lambda^2}{\partial \bar{X}^\theta} = 0, \qquad \gamma = 1, \ldots, n. \tag{1}$$

In addition if the harmonic map is diffeomorphic in the overlap region then the above system is uniformly elliptic.

Proof The action of the Laplace–Beltrami operator on a scalar function $\phi(x^1, \ldots, x^n)$ on M^n is given by

$$\Delta\phi = g^{\alpha\beta}\left[\frac{\partial^2\phi}{\partial x^\alpha \partial x^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial\phi}{\partial x^\gamma}\right]. \tag{2}$$

in the x^α coordinates. It is well known that these connection coefficients obey the transformation laws.

$$\Gamma^\gamma_{\alpha\beta} = \frac{\partial x^\gamma}{\partial \bar{X}^\theta}\frac{\partial^2 \bar{X}^\theta}{\partial x^\alpha \partial x^\beta} + \bar{\Gamma}^\mu_{\theta\sigma}\frac{\partial \bar{X}^\theta}{\partial x^\alpha}\frac{\partial \bar{X}^\sigma}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \bar{X}^\mu} \tag{3}$$

where $\bar{\Gamma}^\mu_{\theta\sigma}$ is the Christoffel Symbol of the second kind for the natural coordinate system $\bar{X}^\alpha$.

Suppose we choose $\phi = x^\theta$ (one of the coordinate functions) then equation (2) becomes

$$\Delta x^\theta = -g^{\alpha\beta}\Gamma^\theta_{\alpha\beta}.$$

Substituting from equation (3) and noticing the fact that $g^{\alpha\beta}$ is a tensor of type (0, 2) we get,

$$\Delta x^\theta = -g^{\alpha\beta}\frac{\partial^2 \bar{X}^\gamma}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\theta}{\partial \bar{X}^\gamma} - \bar{\Gamma}^\mu_{\gamma\sigma}\bar{g}^{\gamma\sigma}\frac{\partial x^\theta}{\partial \bar{X}^\mu}.$$

That is

$$\frac{\partial \bar{X}^\sigma}{\partial x^\theta}\Delta x^\theta = -g^{\alpha\beta}\frac{\partial^2 \bar{X}^\sigma}{\partial x^\alpha \partial x^\beta} - \bar{\Gamma}^\sigma_{\gamma\theta}\bar{g}^{\gamma\theta}$$

now imposing $\Delta x^\theta = 0$ for $\theta = 1, \ldots, n$ gives

$$g^{\alpha\beta}\frac{\partial^2 \bar{X}^\sigma}{\partial x^\alpha \partial x^\beta} + \bar{\Gamma}^\sigma_{\gamma\theta}\bar{g}^{\gamma\theta} = 0 \qquad \text{for} \qquad \sigma = 1, \ldots, n. \tag{4}$$

Since $\bar{g}_{\alpha\beta} = \lambda^2\delta_{\alpha\beta}$ we have

$$\bar{g}^{\alpha\beta}\bar{\Gamma}^\theta_{\alpha\beta} = -\frac{(n-2)}{2}\cdot\frac{1}{\lambda^4}\cdot\delta^{\theta\gamma}\frac{\partial\lambda^2}{\partial \bar{X}^\gamma}. \tag{5}$$

Combining (4) and (5) we get equation (1).

We now note that the coefficient of the principal part of the system of second-order partial-differential equations (1) is the symmetric metric $g^{\alpha\beta}$.

$$\det\{g^{\alpha\beta}\} = \{\det \bar{g}^{\alpha\beta}\}\cdot\left\{\det\left(\frac{\partial x^\theta}{\partial \bar{X}^\mu}\right)\right\}^2 = \frac{1}{\lambda^{2n}}\cdot\left\{\det\left(\partial x^\theta \partial \bar{X}^\mu\right)\right\}^2.$$

Thus if the transformation from X^{μ} to x^{α} is diffeomorphic then the partial-differential equations are uniformly elliptic. This proves the theorem.

Notice that if the manifold is Euclidean or if it is an analytic two-dimensional manifold then the second term in equation (1) vanishes. Analyticity is required in order to find a global family of isothermal coordinates. For example, isothermal coordinates $\bar{x}^{\alpha}$ on a portion of a spherical surface can be obtained by the sterographic projection $\bar{x}^1 + i\bar{x}^2 = \tan\psi/2\cdot\exp(i\theta)$ where θ and ψ are spherical polar coordinates. For simple surfaces isothermal coordinates are given in [4].

We will now describe another situation to which the theory developed in this paper is applicable.

2.3. Surface Construction Using Harmonic Mappings

We consider the problem of generating a coordinate surface in the Euclidean three space E^3 connecting two one-dimensional boundaries (lying in two boundary surfaces) as shown in Figure 2. Let the embedding of the unknown two-dimensional manifold M^2 in E^3 be described by $x^i(\zeta^1, \zeta^2)$, $i = 1, 2, 3$. The tangent vectors are

$$B^i_\alpha := \frac{\partial x^i}{\partial \zeta^\alpha}, \qquad i = 1, 2, 3, \quad \alpha = 1, 2.$$

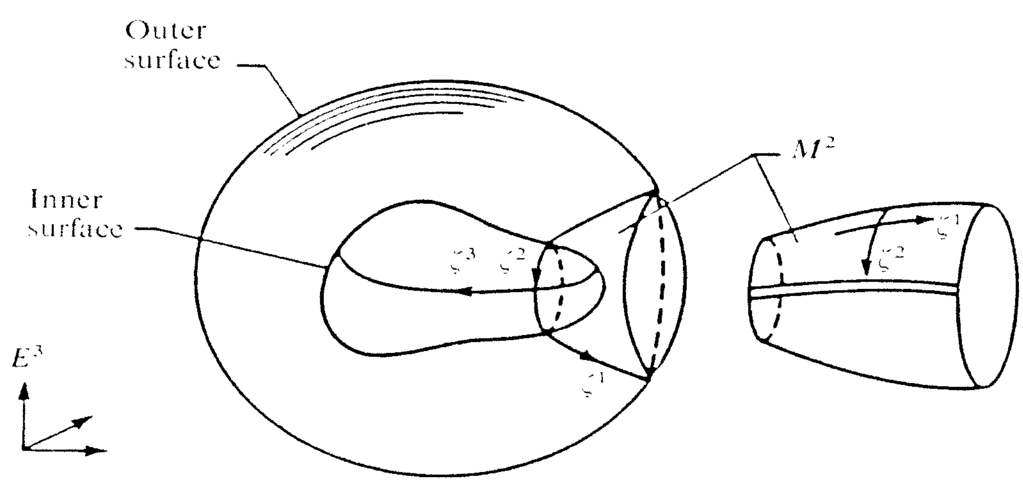

FIGURE 2.

The Gauss equations [4] are given by

$$B^i_{\alpha|\beta} = \Omega_{\alpha\beta} N^i, \qquad i = 1, 2, 3, \quad \alpha = 1, 2,$$

where "|" denotes surface covariant differentiation, $\Omega_{\alpha\beta}$ is the second fundamental form, and N^i is the normal to the manifold. Multiplying by $g^{\alpha\beta}$ yields

$$g^{\alpha\beta} \frac{\partial^2 x^i}{\partial \zeta^\alpha \partial \zeta^\beta} - g^{\alpha\beta} \Gamma^\lambda_{\alpha\beta} B^i_\lambda + \Gamma^i_{hk} B^h_\alpha B^k_\beta g^{\alpha\beta} = (k_1 + k_2) N^i$$

where $\Gamma^\lambda_{\alpha\beta}$ and Γ^i_{hk} are respectively the connection coefficients of M^2 and E^3. Also $(k_1 + k_2)$ is twice the mean curvature of M^2. If x^i are Cartesian coordinates then $\Gamma^i_{hk} = 0$ and recalling our earlier development we obtain

$$g^{\alpha\beta} \frac{\partial^2 x^i}{\partial \zeta^\alpha \partial \zeta^\beta} + B^i_\lambda \Delta \zeta^\lambda = (k_1 + k_2) N^i,$$

where Δ is the Laplace–Beltrami operator on M^2.

If we impose $\Delta \zeta^\lambda = 0$ for $\lambda = 1, 2$, we get

$$g^{\alpha\beta} \frac{\partial^2 x^i}{\partial \zeta^\alpha \partial \zeta^\beta} = (k_1 + k_2) N^i, \qquad i = 1, 2, 3$$

which could be used as generating equations for M^2. This technique has been exploited computationally by Warsi [12], [13].

2.4. Unified Formulation

In this section we will attempt to unify the harmonic mapping methods for domains in curved surfaces and Euclidean spaces (including multiply connected domains) to obtain a mapping problem of simply connected domains in Euclidean spaces. If the manifold M^n is analytic and conformally flat then it may be possible to find a global family of isothermal coordinates for a coordinate patch that contains Ω. Existence of such a global coordinate system would indicate that the domain Ω in M^n can be mapped conformally to the Euclidean n-space R^n. If the manifold is two dimensional the uniformization theorem (stated below) provides the existence theory for such coordinates. In order to include multiply connected domains, we should first construct the universal cover of the manifold which is simply connected.

UNIFORMIZATION THEOREM [8] *Every simply connected Riemann surface is biholomorphically equivalent to either the complex plane C, Riemann sphere $C \cup \{\infty\}$ or the unit disc $\Delta = \{\sigma \in C: |\sigma| < 1\}$.*

Let P be a point on the Riemann surface M^2 and $\hat{P}$ be one of its images on the universal covering $\hat{M}^2$. The projection mapping π is given by $\pi(\hat{P}) = P$ (note π^{-1} is not single valued). Let f map $\hat{M}^2$ to the corresponding canonical domain dictated by the uniformization theorem, then f is univalent, and

$$\sigma = \bar{x}^1 + i\bar{x}^2 = f(\hat{P})$$

or

$$P = (\pi \circ f^{-1})(\sigma) = F(\sigma).$$

Thus $\sigma = F^{-1}(P)$ is the uniformizer for the Riemann surface M^2. This procedure has provided us with a global family of isothermal coordinates $(\bar{x}^1, \bar{x}^2)$ on the manifold M^2. We will now express the Laplacian in these uniformizing variables,

$$\Delta = \frac{1}{\lambda^2}\left[\frac{\partial^2}{\partial \bar{x}^\alpha \, \partial \bar{x}^\alpha}\right], \qquad \lambda = \lambda(\bar{x}^1, \bar{x}^2).$$

Hence the harmonic grid generations of all the cases discussed above reduce to analyzing the following harmonic mapping problem for domains in Euclidean spaces. In what follows we will prove a result that is more or less sufficient to establish that harmonic grid generation provides a globally one-to-one mapping. It is more because part of the result is valid for high dimensions and nonharmonic mappings. It is less because the smoothness on the data required at $\partial\Omega_0$ may be violated at a finite number of points in a practical computation.

3. THE HARMONIC MAPPING THEOREM

Let Ω_0 and Ω_1 be two simply connected homeomorphic regions in R^n with compact closure. Furthermore, assume that Ω_1 is convex. We can define a mapping T from Ω_0 to R^n by specifying a homeomorphism,

$$\phi: \partial\Omega_0 \to \partial\Omega_1$$

and then solving the boundary value problem

$$\Delta T = 0$$

$$T(\sigma) = \phi(\sigma) \qquad \text{if} \quad \sigma \in \partial\Omega_0.$$

Let $T(\sigma) = (x^1(\sigma), \ldots, x^n(\sigma))$. If x^i are obtained as described above, we say they were generated by the harmonic mapping technique. We now state the central theorem of this paper. Namely that the harmonic mapping technique produces a regular coordinate mapping from Ω_0 to Ω_1.

THEOREM 2 *Let $T = (x^1, x^2)$ be generated by the harmonic mapping technique and let the regions Ω_0 and Ω_1 be as above in R^2. We suppose that harmonic conjugates x^{1*} and x^{2*} exist and are continuous on $\bar{\Omega}_0$. Then T is a diffeomorphism from Ω_0 onto Ω_1. In particular, for every* $\sigma \in \Omega_0$, $\det(T'(\sigma)) \neq 0$.

Proof We will establish the proof using three lemmas. One observes that the first and the third lemma are valid for mappings in R^n.

LEMMA 3.1 *Let $T = (x^1, \ldots, x^n)$ be generated by harmonic mapping and let the regions Ω_0 and Ω_1 be as defined above in R^n. Then T maps $\bar{\Omega}_0$ onto $\bar{\Omega}_1$.*

Proof Note first that $T(\Omega_0) \subset \Omega_1$. This follows by noting that since $\bar{\Omega}_1$ is convex, we can express $\bar{\Omega}_1$ as an intersection of closed half spaces,

$$\bar{\Omega}_1 = \bigcap_{\alpha \in R^n} \{\mathbf{x} \in R^n \colon \boldsymbol{\alpha} \cdot \mathbf{x} \leqslant C_{\boldsymbol{\alpha}}\}.$$

Now for each $\boldsymbol{\alpha} \in R^n$, $\boldsymbol{\alpha} \cdot T = \sum_{i=1}^n \alpha_i x^i$ is harmonic, and hence by the maximum principle we have for all $\sigma \in \bar{\Omega}_0$,

$$\boldsymbol{\alpha} \cdot T(\sigma) \leqslant C_{\boldsymbol{\alpha}}.$$

Thus $T(\Omega_0) \subset \Omega_1$. However, since an $(n-1)$-sphere cannot be a retract of an n-ball (by a corollary to Brower's theorem [2, p. 341]) we conclude that T must map $\bar{\Omega}_0$ onto $\bar{\Omega}_1$.

LEMMA 3.2 *Let $T = (x^1, x^2)$ be generated by the harmonic mapping technique and suppose Ω_0 and Ω_1 be as above in R^2. Then* $\det(T'(\sigma)) \neq 0$ *for all $\sigma \in \Omega_0$.*

Proof Suppose to the contrary that there is $\sigma_0 \in \Omega_0$ so that

$$\det(T'(\sigma)) = \det(\nabla x^1, \nabla x^2)(\sigma_0) = 0.$$

Hence we see that $\nabla x^1(\sigma_0)$ and $\nabla x^2(\sigma_0)$ must be linearly dependent.

This means there is a nonzero vector (α, β) so that,

$$(\alpha \nabla x^1 + \beta \nabla x^2)(\sigma_0) = (0, 0).$$

We conclude that the analytic function Z defined by

$$Z(\sigma) := W(\sigma) - W(\sigma_0),$$

$$W(\sigma) := [\alpha x^1 + \beta x^2 + i(\alpha x^{1*} + \beta x^{2*})](\sigma_0)$$

has at least a double zero at σ_0 and hence is m to 1 in a deleted neighborhood of σ_0, $m \geqslant 2$. Furthermore, note that for $\sigma \in \partial\Omega_0$,

$$\operatorname{Re}(Z(\sigma)) = (\alpha x^1 + \beta x^2)(\sigma) - P_0,$$

and since as σ traverses $\partial\Omega_0$ counterclockwise $(x^1(\sigma), x^2(\sigma))$ traverses the boundary of a convex set precisely once, we see that $\operatorname{Re}(Z)$ may have no more than two sign changes. Hence the curve $\sigma \to Z(\sigma)$ for $\sigma \in \partial\Omega_0$ has index 0 or 1 with respect to the origin provided $0 \notin Z(\partial\Omega_0)$. This, of course, contradicts the argument principle which guarantees that the index is m or larger. If $0 \in Z(\partial\Omega_0)$ then we may find an arbitrarily small complex number η_0 so that $0 \notin Z(\partial\Omega_0) + \eta_0$. But then we can make the same argument with $Z + \eta_0$. This contradiction yields the results that $T'(\sigma)$ is invertible for all $\sigma \in \Omega_0$ and the lemma is proven. We have thus proven that T is locally one-to-one and onto.

LEMMA 3.3 *Let $\bar{\Omega}_0, \bar{\Omega}_1 \in R^n$. We assume that both sets are homeomorphic images of $[0, 1]^n$ (the closed n-cube). As before, Ω_0 and Ω_1 denote respectively the interiors of $\bar{\Omega}_0$ and $\bar{\Omega}_1$. Suppose $T: \bar{\Omega}_0 \to \bar{\Omega}_1$ is a differentiable transformation satisfying*

(i) *T is homeomorphic from $\partial\Omega_0$ onto $\partial\Omega_1$,*
(ii) $\det(T^1(x)) \neq 0$ *if $x \in \Omega_0$.*

Then $T(\Omega_0) = \Omega_1$ and T is a homeomorphism from $\bar{\Omega}_0$ to $\bar{\Omega}_1$.

Proof In order to prove the lemma, we must recall some facts concerning topological degree. Let C be an open set in R^n with compact closure $\bar{C}$. Let $F: \bar{C} \to R^n$ be a continuous transformation. Then for any $y \notin F(\partial C)$, the integer $\deg(F, C, y)$ is defined by

$$\deg(F, C, y) := \sum_{\substack{x \in C \\ F(x) = y}} \operatorname{sign}(\det F'(x))$$

provided $F'(x)$ is invertible for all x satisfying $F(x) = y$. In general, the degree is defined in terms of continuous functions and is invariant under homotopy provided no solutions are introduced at the boundary. In fact, one can show that the degree depends only on the boundary values (the so-called Boundary Value Theorem [7]). Furthermore, it is known that [2, Chapter XVI] the degree of a map which is a homomorphism on the boundary of a sphere is either $+1$ or -1. Since this is the case in condition (i) of our lemma (we actually have a homeomorphic image of a sphere) we conclude that for any $y \in \Omega_1$,

$$\det(T, \bar{\Omega}_0, y) = \pm 1 = \sum_{Tx=y} \operatorname{sign} \det(T'(x)), \qquad x \in \Omega_0.$$

But from (ii) we conclude that the sign of $\det(T'(x))$ is constant on Ω_0 and hence for each $y \in \Omega_1$ there is exactly one $x \in \Omega_0$ that $T(x) = y$. It follows that T is bijective on Ω_0 and hence by standard results [2, p. 226], T is a homeomorphism. This completes the proof of this lemma and Theorem 2.

We finally remark that if x^1 and x^2 are in $C^1(\bar{\Omega}_0)$ and if every point σ on $\partial\Omega_0$ is accessible by a rectifiable curve γ starting from some fixed $\sigma_0 \in \Omega_0$ then x^{1*} and x^{2*} are in $C(\bar{\Omega}_0)$. This can be easily seen by recalling

$$x^{i*}(\sigma) = x^{i*}(\sigma_0) + \int_\gamma dx^{i*} = x^{i*}(\sigma_0) + \int_\gamma -\frac{\partial x^i}{\partial y}\,dx + \frac{\partial x^i}{\partial x}\,dy$$

for $i = 1, 2$, where we have used the Cauchy–Riemann equations for the last equality.

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