Theory of Harmonic Grid Generation—II

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Abstract In this paper we will continue the study of harmonic grid generation methods initiated in the earlier paper. New aspects of the results are the regularity properties and the underlying variational principle.

KEY WORDS: Numerical grid generation, harmonic maps, computational physics.

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1 INTRODUCTION

Harmonic mapping was one of the earliest and is perhaps the most widely used grid generation technique in computational physics. In [8] a mathematical study of this method was presented. In this paper we will further elaborate the underlying mathematical structure of this method and sharpen some of the results. The mathematical analysis presented in [8] is very general and is applicable to nonsmooth domains in Riemannian manifolds as well. In this paper, however, we will restrict ourselves to smooth domains in Euclidean two and three spaces. New features explored in this paper are the variational formulation and the concept of duality.

2 VARIATIONAL FORMULATION

We will begin with the definition of grid generating maps.

Definition 2.1 Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with Lipschitz boundary \( \partial \Omega \) and let \( \Omega_1 \subset \mathbb{R}^n \) be a bounded convex open set with boundary \( \partial \Omega_1 \). Suppose there exists a continuous transformation \( \alpha : \Omega \to \mathbb{R}^n \) such that

\[
\alpha|_{\partial \Omega} = \alpha_b : \partial \Omega \to \partial \Omega_1
\]
is a specified homeomorphism onto. Then \( \alpha \) is called a grid generating transform if
\[
\alpha : \Omega \rightarrow \Omega_1
\]
is a homeomorphism on to.
\( \Omega \) and \( \Omega_1 \) are often called the physical and the computational domains respectively. Grid generating transforms belong to a general class of continuous maps known as regular maps[1] which map the boundary of the domain on to the boundary of the image.

Remarks:
(I) \( \Omega \) and \( \Omega_1 \) are called physical and computational domains respectively (see Figure 1.1).
(II) \( \Omega_1 \) is usually (in computational mechanics literature) taken as a rectangle.
(III) The above definition can be generalized (as in [8]) to include grid generation on surfaces. For this purpose we need to replace \( \mathbb{R}^n \) in the above definition by a Riemannian manifold \( M^n \).
(IV) The boundary homeomorphism \( \partial \Omega \rightarrow \partial \Omega_1 \) is constructed by the computational scientist to meet his or her requirements.

At this stage we have a very general grid generation method. Let us now establish a result for such methods and identify the basic mathematical properties.
needed by any plausible grid generation procedure. For this purpose, we will prove a fundamental theorem for general grid generation methods.

Theorem 2.1 Let \( \alpha : \Omega \to \mathbb{R}^n \) be a continuous transformation such that \( \alpha \) is a homeomorphism onto \( \partial \Omega \to \partial \Omega_1 \). Let

(I) \( \alpha \) maps \( \Omega \) onto \( \Omega_1 \),

(II) \( \text{grad } \alpha \in L^\infty(\Omega) \) and

(III) \( \det (\nabla \alpha) \neq 0 \), in \( \Omega \).

Then \( \alpha \) is a grid generating transform.

Proof We only need to show that the transformation is one to one. This is accomplished using an integral representation of the degree of the map which requires weaker regularity. Recall that for any \( y = \alpha(x) \) with \( y \not\in \partial \Omega \), the degree of the map \( \alpha \) is defined as [7],

\[
D_\alpha(y; \Omega) = \int_{\Omega} f_\epsilon(\alpha(x)) \det (\nabla \alpha(x)) \, dx
\]

where \( f_\epsilon(\cdot) : \mathbb{R}^n \to \mathbb{R} \) is a family of continuous transformations such that

\[
\int_{B(y;\epsilon)} f_\epsilon(y) \, dy = 1
\]

where the support of \( \{ f_\epsilon \} = B(y; \epsilon) \) is a ball of sufficiently small radius \( \epsilon \) centered at \( y \). If the sign of the determinant of \( \nabla \alpha \) is constant almost everywhere in \( \Omega \) then

\[
D_\alpha(y; \Omega) = \pm \text{Number of solutions } x \in \Omega \text{ for a given } y = \alpha(x).
\]

To see this we will begin with a continuously differentiable map \( \alpha \) and suppose that \( x_1, \ldots, x_m \in \Omega \) be the elements of \( \alpha^{-1}(y) \). For \( \epsilon > 0 \) small enough, \( \exists \) neighborhoods \( N(x_i; \epsilon) \subset \Omega, i = 1, \ldots, m \) such that

\[
\alpha : N(x_i; \epsilon) \to B(y, \epsilon) \text{ homeomorphically.}
\]

Hence,

\[
\int_{\Omega} f_\epsilon(\alpha(x)) \det (\nabla \alpha(x)) \, dx = \sum_{i=1}^{m} \int_{N(x_i; \epsilon)} f_\epsilon(\alpha(x)) \det (\nabla \alpha(x)) \, dx,
\]
since \( f_i(\alpha(x)) \) is zero outside \( \bigcup_{i=1}^{m} N(x; \varepsilon) \). Thus changing the variables of integration from \( x \) to \( y \) and noting that the sign of \( \det \{ \nabla \alpha(x) \} \) is same in each \( N(x; \varepsilon) \), we get,

\[
D_\alpha(y; \Omega) = \sum_{i=1}^{m} \text{sign} \det \{ \nabla \alpha(x_i) \} \int_{B(y;\varepsilon)} f_i(y) \, dy
\]

\[
= \sum_{i=1}^{m} \text{sign} \det \{ \nabla \alpha(x_i) \}.
\]

Hence, if the sign of the determinant is constant in \( \Omega \), then,

\[
D_\alpha(y; \Omega) = \pm m.
\]

In order to justify these arguments to the case where \( \alpha \) is not \( C^1 \), we only need to construct an approximating sequence \( \alpha_k \) of \( C^1 \) maps such that \( \det \{ \nabla \alpha_k \} \rightarrow \det \{ \nabla \alpha \} \) weakly in \( L^1(\Omega) \).

The above result means that the degree of the map \( \alpha \) in our case is equal to the number of points in \( \Omega \) which are mapped to the point \( y \in \Omega_1 \) under the transformation \( \alpha \). Let us now provide arguments to show that the degree is equal to \( \pm 1 \). This results comes from a general theorem for regular maps [4].

**Proposition 2.1** Let \( \beta \) be a continuous map such that,

\[
(I) \; \beta : B^n \rightarrow B^n \text{ where } B^n \text{ is the n-ball.}
\]

\[
(II) \; \beta_\partial : \partial B^n \rightarrow \partial B^n.
\]

Then

\[
\text{Degree of } \beta_\partial = \text{Degree of } \beta.
\]

Moreover, if \( \beta_\partial : \partial B^n \rightarrow \partial B^n \) is a homeomorphism onto then the Degree of \( \beta_\partial = \pm 1 \).

Note that this is precisely the situation we have. Therefore, the degree of the map \( \alpha \) is the same as the degree of its restriction to the boundary. However, it is given that the boundary map is a homeomorphism onto and hence the degree of \( \alpha \) (as well as its restriction to \( \partial \Omega \)) is simply \( \pm 1 \).
This theorem provides important informations regarding the selection of grid generating procedures. As we remarked earlier, the boundary homeomorphism $\alpha_b : \partial\Omega \to \partial\Omega_1$ is prescribed by the computational scientist. It is therefore necessary to make sure that the properties (I), (II) and (III) stated in the theorem (2.1) are satisfied by the chosen grid generation method.

A grid generating transform $\alpha$ is harmonic if $\Delta \alpha = 0$. Here $\Delta$ denotes the Laplacian operator and $\alpha = (\alpha_1, \cdots, \alpha_n)$.

Our goal is to find a harmonic grid generating map for a given domain $\Omega$. We will only be interested in the cases $n = 2$ and $3$ which are of practical interest.

Let us begin with the following well known orthogonal decomposition[3] of $L^2(\Omega)$.

$$L^2(\Omega) = \mathcal{H}(\Omega) \oplus \mathcal{G}(\Omega)$$

where

$$\mathcal{H}(\Omega) = \{ u \in L^2(\Omega); \nabla \cdot u = 0 \}$$

and

$$\mathcal{G}(\Omega) = \{ u \in L^2(\Omega); u = \nabla \alpha, \alpha \in H^1_0(\Omega) \}.$$

Here $H^1_0(\Omega)$ denotes the Sobolev space of square integrable vectorfields (or tensorfields) with square integrable distributional derivatives and zero boundary values.

Let us define the normal trace operator $\gamma_\nu$ as

$$\gamma_\nu u = u \cdot n|_{\partial\Omega} \ \forall u \in C(\bar{\Omega}).$$

The following result is a slightly specialized version of a theorem in [10].

**Lemma 2.1** Let $\Omega \subset \mathbb{R}^n$, with $\partial\Omega \in C^2$. Then the trace operator $\gamma_\nu : \mathcal{H}(\Omega) \to H^{-1/2}(\partial\Omega)$ continuously.

Here $H^{-1/2}(\partial\Omega)$ is the dual of the Sobolev space $H^{1/2}(\partial\Omega)$.

The proof is simple and we will outline it below.

**Proof:**

Recall that the trace operator $\gamma_0$ defined by

$$\gamma_0 u = u|_{\partial\Omega}, \ \forall u \in C(\bar{\Omega})$$

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can be extended as $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ continuously. Moreover the right inverse of this operator $I_\Omega \in \mathcal{L}(H^{1/2}(\partial\Omega); H^1(\Omega))$. Now let the vectorfield $u \in L^2(\Omega)$ be given. Let us consider

$$\chi_u(\phi) = \int_\Omega u \cdot \nabla(l_\Omega \phi) dx, \quad \forall \phi \in H^{1/2}(\partial\Omega)$$

We have by Schwartz inequality

$$|\int_\Omega u \cdot \nabla(l_\Omega \phi) dx| \leq \|u\|_{L^2(\Omega)} \|l_\Omega \phi\|_{H^{1/2}(\partial\Omega)}$$

Hence

$$|\chi_u(\phi)| \leq C_0 \|u\|_{L^2(\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)}, \quad \forall \phi \in H^{1/2}(\partial\Omega)$$

That is for each $u \in L^2(\Omega)$ the linear map $\chi_u(\cdot) : H^{1/2}(\partial\Omega) \rightarrow \mathbb{R}$ continuously. Hence by Riesz representation theorem there exists $u \in H^{-1/2}(\partial\Omega)$ such that

$$\int_\Omega u \cdot \nabla(l_\Omega \phi) dx = \langle u, \gamma_0 \phi \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}$$

To interpret the element $\gamma_\nu u$ we take $u \in C^1(\Omega)$ such that $\nabla \cdot u = 0$ and $\phi \in C^1(\Omega)$. Then integrating the above integral by parts we get

$$\langle \gamma_\nu u, \gamma_0 \phi \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} = \int_{\partial\Omega} u \cdot n \phi ds$$

Thus $\gamma_\nu u = u \cdot n|_{\partial\Omega}, \forall u \in C^1(\Omega)$.

The following result is a special case of a theorem in [9].

**Lemma 2.2** Let $\Omega$ be simply connected with boundary $\partial\Omega$ of class $C^r, r \geq m + 2, m \geq 0$. Then

$$\text{curl } H^{m+1}(\Omega) = H^m(\Omega) \cap \mathcal{H}(\Omega)$$

and curl is an isomorphism from $H^{m+1}(\Omega)$ on to $H^m(\Omega) \cap \mathcal{H}(\Omega)$.

Let us denote the gradient operator by

$$\Lambda = \text{grad} \quad \text{in } \mathcal{D}(\Omega)' ,$$

where $\mathcal{D}(\Omega)'$ is the space of distributions (dual to the space of test functions $\mathcal{D}(\Omega)$).
Lemma 2.3 \( \Lambda \in \mathcal{L}(H^1(\Omega); L^2(\Omega)) \) and its transpose \( \Lambda^* \in \mathcal{L}(L^2(\Omega); (H^1(\Omega))') \).

Moreover, if we restrict \( \Lambda \) to \( H^1_0(\Omega) \) then denoting its transpose by \( \Lambda^*_1 \in \mathcal{L}(L^2(\Omega); H^{-1}(\Omega)) \), we get \( \Lambda^*_1 = \text{div} \) in \( \mathcal{D}(\Omega)' \).

Proof

Let us note that by Schwartz inequality we have

\[
| \int_\Omega \nabla \alpha \cdot Q dx | \leq \| \alpha \|_{H^1(\Omega)} \| Q \|_{L^2(\Omega)}, \forall \alpha \in H^1(\Omega) \text{ and } \forall Q \in L^2(\Omega)
\]

Hence for a given tensorfield \( Q \in L^2(\Omega) \),

\[
| \int_\Omega \nabla \alpha \cdot Q dx | \leq C \| \alpha \|_{H^1(\Omega)}, \forall \alpha \in H^1(\Omega)
\]

Thus by Riesz representation theorem \( \exists \) a vectorfield \( \beta^* \in (H^1(\Omega))' \) such that

\[
\int_\Omega \nabla \alpha \cdot Q dx = \langle \beta^*, \alpha \rangle_{(H^1(\Omega))' \times H^1(\Omega)}
\]

This defines a unique operator \( \Lambda^* \in \mathcal{L}(L^2(\Omega); (H^1(\Omega))') \) such that \( \beta^* = \Lambda^* Q \) and

\[
(Q, \Lambda \alpha)_{L^2(\Omega)} = \langle \Lambda^* Q, \alpha \rangle_{(H^1(\Omega))' \times H^1(\Omega)}, \forall \alpha \in H^1(\Omega) \text{ and } \forall Q \in L^2(\Omega)
\]

Since the above estimate on the integral holds also \( \forall \alpha \in H^1_0(\Omega) \), we can define \( \Lambda^*_1 \in \mathcal{L}(L^2(\Omega); H^{-1}(\Omega)) \) such that

\[
(Q, \Lambda \alpha)_{L^2(\Omega)} = \langle \Lambda^*_1 Q, \alpha \rangle_{H^{-1}(\Omega) \times H^1(\Omega)}, \forall \alpha \in H^1_0(\Omega) \text{ and } \forall Q \in L^2(\Omega)
\]

Now note that integration by parts gives

\[
\int_\Omega \nabla \alpha \cdot Q dx = - \int_\Omega \alpha \cdot \text{div} Q dx, \ \forall \alpha \in \mathcal{D}(\Omega) \text{ and } \forall Q \in \mathcal{D}(\Omega)
\]

Hence \( \Lambda^*_1 = - \text{div} \) in \( \mathcal{D}(\Omega)' \). Note however that

\[
\int_\Omega \nabla \alpha \cdot Q dx = - \int_\Omega \alpha \cdot \text{div} Q dx + \int_{\partial \Omega} \alpha \cdot Q \cdot n dS, \ \forall \alpha \in C^1(\bar{\Omega}) \text{ and } \forall Q \in C^1(\bar{\Omega})
\]

This explains the difference between \( \Lambda^*_1 \) and \( \Lambda^* \).
We will now provide a variational formulation for our grid generation problem. Let us denote by $A \subset H^1(\Omega)$ the closed convex subset defined as

$$A = \alpha_0 + H^1_0(\Omega),$$

where

$$\alpha_0 \in H^1(\Omega)$$

be such that

$$\alpha_0|_{\partial \Omega} = g \in H^{1/2}(\partial \Omega)$$

with a given boundary distribution of a vectorfield $g$.

Let us denote by $\delta(\alpha|A)$ the indicator function defined as

$$\delta(\alpha|A) = \begin{cases} 0 & \text{if } \alpha \in A \\ +\infty & \text{otherwise} \end{cases}$$

**Problem 2.1 Primal variational problem**

*Find a vectorfield $\alpha : \Omega \to \mathbb{R}^n$ such that $\alpha \in H^1(\Omega)$ and

$$J(\alpha) = \frac{1}{2}\|\nabla \alpha\|^2_{L^2(\Omega)} + \delta(\alpha|A) \to \inf$$

Here $\nabla \alpha$ is written componentwise as

$$\frac{\partial \alpha_i}{\partial x_j}, \ i,j = 1, \ldots, n.$$ and

$$\|\nabla \alpha\|^2_{L^2(\Omega)} = \int_{\Omega} |\nabla \alpha|^2 \, dx = \sum_{i=1}^n \int_{\Omega} \left| \frac{\partial \alpha_i}{\partial x_j} \right|^2 \, dx$$

Note that in this variational formulation we are looking for a solution such that $\alpha - \alpha_0 \in G(\Omega)$.

Let us now consider the dual formulation.

**Problem 2.2 Dual variational problem**

*Find a tensorfield $Q^* : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ such that $Q^* \in \mathcal{H}(\Omega)$ and

$$J^*(Q^*) = \gamma_0 \langle Q^*, g \rangle_{H^{-1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)} - \frac{1}{2}\|Q^*\|^2_{L^2(\Omega)} \to \sup$$

(2)
Here $Q^* \in \mathcal{H}(\Omega)$ implies that
\[
\sum_{i,j=1}^{n} \int_{\Omega} (Q^*_{ij})^2 dx < \infty \text{ and }
\]
\[
[\nabla \cdot Q^*]_i = \frac{\partial Q^*_{ij}}{\partial x^j} = 0, \quad i = 1, \cdots, n.
\]
Moreover, the duality pairing
\[
\langle \gamma \cdot Q^*, g \rangle_{H^{-1}(\Omega) \times H^1(\Omega)} = \sum_{i,j=1}^{n} \int_{\Omega} Q^*_{ij} n_j g_i dS
\]
Let us now verify that this dual variational principle can be derived from the primal problem using the concept of polar functions [2]. We will begin with the following perturbed variational problem obtained from the primal problem.

Let $Q \in L^2(\Omega)$ be a tensorfield. Let us find $\alpha \in H^1(\Omega)$ such that
\[
\Phi(\alpha, Q) = \frac{1}{2} \|\Lambda \alpha + Q\|_{L^2(\Omega)}^2 + \delta(\alpha|\mathcal{A}) \to \inf
\]
We define the value function $V(\cdot): L^2(\Omega) \to \mathbb{R}$ as
\[
V(Q) = \inf_{\alpha \in H^1(\Omega)} \Phi(\alpha, Q)
\]
Note that $V(0)$ corresponds to the primal problem. We will now note that the polar function corresponds to $\Phi(\cdot, \cdot)$ is
\[
\Phi^*(\alpha^*, Q^*) = \sup_{\alpha \in H^1(\Omega)} \sup_{Q \in L^2(\Omega)} \{ < \alpha^*, \alpha >_{H^1(\Omega) \times H^1(\Omega)} + (Q^*, Q)_{L^2(\Omega)} - \Phi(\alpha, Q) \}
\]
We will show below that the dual problem is actually that of finding $Q^* \in L^2(\Omega)$ such
\[
- \Phi^*(0, Q^*) \to \sup
\]
Let us first establish the following relationship between $\Phi^*(0, Q^*)$ and the bidual of the value function $V^{**}(\cdot)$:
\[
V^{**}(0) = \sup_{Q^* \in L^2(\Omega)} \{ -\Phi^*(0, Q^*) \}
\]
In fact if we consider
\[
V^*(Q^*) = \sup_{Q \in L^2(\Omega)} \{ (Q^*, Q)_{L^2(\Omega)} - V(Q) \}
\]
and

\[ V^*(Q^*) = \sup_{Q^* \in L^2(\Omega)} \{ (Q^*, Q^*)_{L^2(\Omega)} - V^*(Q^*) \} \]

Then

\[ V^*(0) = \sup_{Q^* \in L^2(\Omega)} \{ -V^*(Q^*) \} \]

\[ = \sup_{Q^* \in L^2(\Omega)} \left\{ \sup_{Q \in L^2(\Omega)} \left\{ (Q^*, Q)_{L^2(\Omega)} - V(Q) \right\} \right\} \]

\[ = \sup_{Q^* \in L^2(\Omega)} \left\{ \sup_{Q \in L^2(\Omega)} \left\{ (Q^*, Q)_{L^2(\Omega)} - \inf_{\alpha \in H^1(\Omega)} \Phi(\alpha, Q) \right\} \right\} \]

\[ = \sup_{Q^* \in L^2(\Omega)} \left\{ \sup_{Q \in L^2(\Omega)} \left\{ (Q^*, Q)_{L^2(\Omega)} - \Phi(\alpha, Q) \right\} \right\} \]

Hence

\[ V^*(0) = \sup_{Q^* \in L^2(\Omega)} \{ -\Phi^*(0, Q^*) \} \]

Which verifies (6). We will now consider

\[ \Phi^*(0, Q^*) = \sup_{Q \in L^2(\Omega)} \sup_{\alpha \in H^1(\Omega)} \{ (Q^*, Q)_{L^2(\Omega)} - \frac{1}{2} \| \Lambda \alpha + Q \|_{L^2(\Omega)}^2 - \delta(\alpha, \mathcal{A}) \} \]

\[ = \sup_{Q \in L^2(\Omega)} \sup_{\alpha \in H^1(\Omega)} \{ (Q^*, Q + \Lambda \alpha)_{L^2(\Omega)} - \frac{1}{2} \| Q + \Lambda \alpha \|_{L^2(\Omega)}^2 - (\Lambda \alpha, Q^*)_{L^2(\Omega)} - \delta(\alpha, \mathcal{A}) \} \]

\[ = \sup_{P \in L^2(\Omega)} \{ (Q^*, P)_{L^2(\Omega)} - \frac{1}{2} \| P \|_{L^2(\Omega)}^2 + \sup_{\alpha \in H^1(\Omega)} \{ -\Lambda^* Q^*, \alpha >_{(H^1(\Omega))'} x_{H^1(\Omega)} - \delta(\alpha, \mathcal{A}) \} \}
\]

\[ = \frac{1}{2} \| Q^* \|_{L^2(\Omega)}^2 + \delta^*(-\Lambda^* Q^* | \mathcal{A}) \]

Where \( \delta^*(-\mathcal{A}) \) is the polar of the indicator function \( \delta(-\mathcal{A}) \). Hence (6) becomes

\[ V^*(0) = \sup_{Q^* \in L^2(\Omega)} \left\{ -\frac{1}{2} \| Q^* \|_{L^2(\Omega)}^2 - \delta^*(-\Lambda^* Q^* | \mathcal{A}) \right\} \]

Let us consider

\[ \delta^*(-\Lambda^* Q^* | \mathcal{A}) = \sup_{\alpha \in \mathcal{A}} < -\Lambda^* Q^*, \alpha >_{(H^1(\Omega))'} x_{H^1(\Omega)} \]

Since \( \alpha = \alpha_0 + \beta \), where \( \alpha_0 \in H^1(\Omega) \) was defined earlier and \( \beta \in H^1_0(\Omega) \) we get

\[ \delta^*(-\Lambda^* Q^* | \mathcal{A}) = < -\Lambda^* Q^*, \alpha_0 >_{(H^1(\Omega))'} x_{H^1(\Omega)} + \sup_{\beta \in H^1_0(\Omega)} < -\Lambda^* Q^*, \beta >_{H^{-1}(\Omega) x H^1_0(\Omega)} \]

Note that

\[ \sup_{\beta \in H^1_0(\Omega)} < -\Lambda^* Q^*, \beta >_{H^{-1}(\Omega) x H^1_0(\Omega)} = \begin{cases} 0 & \text{if } \Lambda^* Q^* = 0 \\ +\infty & \text{otherwise} \end{cases} \]
Hence
\[
\sup_{Q^* \in \mathcal{L}(\Omega)} \{-\delta^*(\Lambda^*Q^* | A)\} = \sup_{Q^* \in \mathcal{L}(\Omega)} \{-<\Lambda^*Q^*, \alpha_0 >_{(H^1(\Omega))^{' \times H^1(\Omega)}} - 0\}
\]

with \(\Lambda^*Q^* = 0\). Thus
\[
V^{**}(0) = \sup_{Q^* \in \mathcal{L}(\Omega)} \left\{-\frac{1}{2}\|Q^*\|^2 + <\Lambda^*Q^*, \alpha_0 >_{(H^1(\Omega))^{' \times H^1(\Omega)}}\}
\]

with \(\Lambda^*Q^* = 0\). Let us further interpret this result.

\[
\Lambda^*Q^* = \text{div } Q^* = 0
\]

Hence, \(Q^* \in \mathcal{H}(\Omega)\) and this implies by lemma 2.1 that \(\gamma_\partial Q^* \in H^{-1/2}(\partial\Omega)\). Thus we can write
\[
<\Lambda^*Q^*, \alpha_0 >_{(H^1(\Omega))^{' \times H^1(\Omega)}} = (Q^*, \Lambda\alpha_0)_{L^2(\Omega)} = \int_{\Omega} Q^* \cdot \nabla \alpha_0 dx
\]

\[
= -\int_{\Omega} \alpha_0 \cdot \text{div } Q^* dx + \int_{\partial\Omega} g \cdot Q^* \cdot n ds
\]

From which we get
\[
<\Lambda^*Q^*, \alpha_0 >_{(H^1(\Omega))^{' \times H^1(\Omega)}} = \int_{\partial\Omega} g \cdot Q^* \cdot n ds = <\gamma_\partial Q^*, g >_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}
\]

Thus
\[
V^{**}(0) = \sup_{Q^* \in \mathcal{H}(\Omega)} \left\{-\frac{1}{2}\|Q^*\|^2 + <\gamma_\partial Q^*, g >_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}\}
\]

(7)

This is precisely the dual problem.

Let us now state

**Theorem 2.2** \(\exists\) unique solutions \(\hat{\alpha}\) and \(\hat{Q}^*\) respectively to the primal problem and to the dual problem such that

\[
\inf_{\alpha \in \mathcal{H}^1(\Omega)} \mathcal{J}(\alpha) = \mathcal{J}(\hat{\alpha}) = \sup_{Q^* \in \mathcal{H}(\Omega)} \mathcal{J}^*(Q^*) = \mathcal{J}^*(\hat{Q}^*)
\]

with \(\hat{Q}^* = \nabla \hat{\alpha}\). The vectorfield \(\hat{\alpha}\) satisfies \(\Delta \hat{\alpha} = 0\) in the sense of distributions and

\[
\gamma_\partial \hat{\alpha} = \hat{\alpha}|_{\partial\Omega} = g \text{ in the sense of trace}
\]

Moreover, \(\exists\) a tensorfield \(\hat{\psi} \in H^1(\Omega)\) such that

\[
\nabla \hat{\alpha} = \text{curl } \hat{\psi} \in \mathcal{H}(\Omega).
\]
Proof Let us first note that the perturbed variational problem (3) has a unique solution $\alpha_Q$ for each $Q \in L^2(\Omega)$ including for $Q = 0$ which corresponds to the primal problem. This is essentially the Direchlet Principle and we simply note the main arguments leading to this result. Since the set $A$ is closed, the epigraph 
\[ \operatorname{epi} \delta(-\cdot|A) = \{(\alpha, a) \in H^1(\Omega) \times R|\delta(\alpha|A) \leq a\} \]

is closed and hence the indicator function $\delta(-\cdot|A) : H^1(\Omega) \to R \cup \{+\infty\}$ is lower semicontinuous. Moreover, this is a convex function since $A$ is convex.

The convexity and lower semicontinuity of $\delta(-\cdot|A)$ imply that this function is also weakly sequentially lower semicontinuous in $H^1(\Omega)$. We therefore conclude that $\forall Q \in L^2(\Omega)$, the function $\Phi(\cdot, Q) : H^1(\Omega) \to R \cup \{+\infty\}$ is convex and weakly sequentially lower semicontinuous. Hence we can deduce by standard compactness arguments that for each $Q \in L^2(\Omega)$, $\exists$ a unique solution $\hat{\alpha}_Q$ such that 
\[ V(Q) = \inf_{\alpha \in H^1(\Omega)} \Phi(\alpha, Q) = \Phi(\hat{\alpha}_Q, Q). \]

This shows that for each $Q \in L^2(\Omega)$ the value function is defined with a unique element $\hat{\alpha}_Q \in A$. Setting $Q = 0$ gives us the existence of a unique solution $\hat{\alpha}$ to the primal problem.

From (4), (6) and (7) we know that in order to establish the coincidence of the primal and the dual problems we need to show that 
\[ V(0) = V^*(0). \]

We will now examine the properties of the value function to deduce this result.

Note that $\Phi(\cdot, \cdot) : H^1(\Omega) \times L^2(\Omega) \to R \cup \{+\infty\}$ is convex and this implies that the value function $V(\cdot) : L^2(\Omega) \to R$ is convex.

Now let us consider a sequence $Q^n \to 0$ weakly in $L^2(\Omega)$. Then, since $\forall n, \hat{\alpha}_Q^n \in A$, we have 
\[ V(Q^n) = \Phi(\hat{\alpha}_Q^n, Q^n) = \frac{1}{2}\|\Lambda \hat{\alpha}_Q^n + Q^n\|_{L^2(\Omega)}. \]

Now, using the estimate 
\[ \|\Lambda \hat{\alpha}_Q^n\|_{L^2(\Omega)} \leq C(\|Q^n\|_{L^2(\Omega)}, \|g\|_{H^1(\Omega)}) \]
and the fact that \( \Lambda \) is a closed operator in \( H^1(\Omega) \), we can conclude that \( \Lambda \mathbf{a}_n \rightarrow \Lambda \alpha \) weakly in \( L^2(\Omega) \). Thus \( \Lambda \mathbf{a}_n + Q^n \rightarrow \Lambda \alpha \) weakly in \( L^2(\Omega) \). We conclude then that,

\[
\mathcal{V}(0) \leq \liminf_{n \to \infty} \mathcal{V}(Q^n)
\]

which establishes the lower semicontinuity of \( \mathcal{V}(\cdot) \) at the origin. This, in combination with the fact that \( \mathcal{V}(\cdot) \) is a convex function implies [2] that

\[
\mathcal{V}(0) = \mathcal{V}^{**}(0)
\]

and hence the primal and the dual problems have the same values:

\[
\inf_{\mathbf{r} \in H^1(\Omega)} \mathcal{J}(\mathbf{r}) = \mathcal{J}(\hat{\mathbf{r}}) = \sup_{Q^* \in \mathcal{H}(\Omega)} \mathcal{J}^*(Q^*) = \mathcal{J}^*(\hat{Q}^*)
\]

Let us now derive an implication of this optimality relationship. We have, for the optimal solutions \( \hat{\mathbf{a}} \in \mathcal{A} \) and \( \hat{Q}^* \in \mathcal{H}(\Omega) \),

\[
\frac{1}{2} \| \Lambda \hat{\mathbf{a}} \|_{L^2(\Omega)}^2 = \frac{1}{2} \| \hat{Q}^* \|_{L^2(\Omega)}^2 + \gamma_\alpha \hat{Q}^*, g >_{H^{-1}(\Omega) \times H^1(\Omega)} \gamma_1 \hat{Q}^*, g >_{H^{-1}(\Omega) \times H^1(\Omega)}
\]

That is

\[
\frac{1}{2} \| \Lambda \mathbf{a} - \hat{Q}^* \|_{L^2(\Omega)}^2 + (\Lambda \mathbf{a}, \hat{Q}^*)_{L^2(\Omega)} - \gamma_\alpha \hat{Q}^*, g >_{H^{-1}(\Omega) \times H^1(\Omega)} = 0.
\] (8)

But

\[
(\Lambda \hat{\mathbf{a}}, \hat{Q}^*)_{L^2(\Omega)} = - \gamma_1 \hat{Q}^*, \hat{\mathbf{a}} >_{H^{-1}(\Omega) \times H^1(\Omega)} + \gamma_\alpha \hat{Q}^*, g >_{H^{-1}(\Omega) \times H^1(\Omega)}
\]

with \( \gamma_1 \hat{Q}^* = \text{div} Q^* = 0 \). Thus (8) becomes

\[
\| \Lambda \hat{\mathbf{a}} - \hat{Q}^* \|_{L^2(\Omega)} = 0
\]

and we conclude that

\[
\Lambda \hat{\mathbf{a}} = \hat{Q}^* \in \mathcal{H}(\Omega)
\] (9)

Now, since \( \mathcal{H}(\Omega) = \text{curl} \ H^1(\Omega) \) by lemma 2.2 we can deduce that \( \exists \) a tensorfield \( \mathbf{\hat{\psi}} \in H^1(\Omega) \) such that

\[
\text{grad} \hat{\mathbf{a}} = \text{curl} \mathbf{\hat{\psi}}
\] (10)
This can also be written in the tensor form as
\[
\frac{\partial \hat{\alpha}_i}{\partial x^j} = \epsilon_{jim} \frac{\partial \hat{\psi}_m}{\partial x^l}, \quad i, j = 1, 2, 3.
\]
Moreover, \( \nabla \hat{\alpha} \in \mathcal{H}(\Omega) \) and hence \( \text{div} \cdot \nabla \hat{\alpha} = 0 \). That is
\[
\Delta \hat{\alpha} = 0.
\]

**Remark:** If \( n = 2 \), we have \( \hat{\psi}_m = \delta_{m3} \hat{\psi}_i \) and then equation (10) becomes,
\[
\frac{\partial \hat{\alpha}_i}{\partial x^j} = \epsilon_{jim} \frac{\partial \hat{\psi}_j}{\partial x^l}, \quad i, j = 1, 2.
\]
This is the same as
\[
\frac{\partial \hat{\alpha}_i}{\partial x^j} = \frac{\partial \hat{\psi}_i}{\partial x^j}, \quad i = 1, 2.
\]
This is the Cauchy-Riemann relationship and hence for \( n = 2 \), \( \hat{\psi}_i \) is the complex conjugate of \( \hat{\alpha}_i \). We also have for \( n = 2 \),
\[
\Delta \hat{\alpha} = \Delta \hat{\psi} = 0.
\]

Let us now state a regularity theorem for the vectorfields \( \hat{\alpha} \) and \( \hat{\psi} \).

**Theorem 2.3** Let \( \partial \Omega \in C^{m+2}, m \geq 2 \) and \( g \in H^{m-1/2}(\partial \Omega) \). Then the optimal solutions have the regularity \( \hat{\alpha}, \hat{\psi} \in H^m(\Omega) \)

**Proof**
First note that \( \hat{\alpha} \) solves
\[
\Delta \hat{\alpha} = 0 \text{ with } \hat{\alpha}|_{\partial \Omega} = g \in H^{m-1/2}(\partial \Omega).
\]
Hence by standard regularity theory of the Dirichlet problem, we have \( \hat{\alpha} \in H^m(\Omega) \). Thus \( \Delta \hat{\alpha} = \text{curl} \hat{\psi} \in H^{m-1}(\Omega) \cap \mathcal{H}(\Omega) \). But \( \text{curl} \) is an isomorphism from \( H^m(\Omega) \) on to \( H^{m-1}(\Omega) \cap \mathcal{H}(\Omega) \). Thus \( \hat{\psi} \in H^m(\Omega) \).
3 HARMONIC GRIDS IN TWO DIMENSIONS

In this section we will present a refined version of the theory presented in [8]. The results in this section would establish that the harmonic map in two dimensions is a grid generating transform. The central result is the following:

Theorem 3.1 Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with class $C^4$ boundary $\partial \Omega$ and let $\Omega_1 \subset \mathbb{R}^2$ be a bounded convex open set. Let the homeomorphism $\alpha : \partial \Omega \rightarrow \partial \Omega_1$ be specified by

$$\alpha|_{\partial \Omega} = g \in H^{3/2}(\partial \Omega).$$

Then the optimal solution $\alpha$ of theorem (2.2) is a homeomorphism from $\tilde{\Omega}$ on to $\tilde{\Omega}_1$.

Proof: Note that when $\alpha \in H^{n/2+1}(\Omega), n \geq 2$, we have $\nabla \alpha \in H^{n/2}(\Omega) \subset L^n(\Omega)$ by Sobolev embedding theorem. The results below verify the conditions (I) and (III) of theorem (2.1).

Lemma 3.1 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\partial \Omega$ of class $C^{n/2+3}$ and let $\Omega_1 \subset \mathbb{R}^n$ be a convex bounded domain. Suppose that the optimal solution $\alpha$ be obtained by specifying the boundary homeomorphism $\alpha : \partial \Omega \rightarrow \partial \Omega_1$:

$$\alpha|_{\partial \Omega} = g \in H^{(n+1)/2}(\partial \Omega).$$

Then $\alpha$ maps $\tilde{\Omega}$ on to $\tilde{\Omega}_1$.

Proof: Note that from the regularity results we have $\alpha \in H^{n/2+1}(\Omega) \subset C(\Omega)$ and hence the weak-maximum principle holds [6]. This result and the fact that $\Omega_1$ is convex imply that $\alpha(\Omega) \subset \Omega_1$. To see this first note that the convex domain $\Omega_1$ can be expressed as an intersection of closed half spaces:

$$\Omega_1 = \bigcap_{\mu \in \mathbb{R}^n} \{ \beta \in \mathbb{R}^n ; \mu \cdot \beta \leq C_\mu \}$$

Now for each $\mu \in \mathbb{R}^n$, $\mu \cdot \alpha \in C(\tilde{\Omega})$ is harmonic and by weak maximum principle we have

$$\mu \cdot \alpha(x) \leq C_\mu, \ \forall x \in \tilde{\Omega}.$$
Thus \( \alpha(\Omega) \subseteq \Omega_1 \). However, since an \((n - 1)\) - sphere \( \partial B^n \) cannot be a retract of an \( n \) - ball \( B^n \) (by a corollary to the Brouwer's theorem [4]) we should have \( \alpha(\Omega) = \Omega_1 \).

\[ \alpha(\Omega) \subseteq \Omega_1. \]

\[ \text{Proposition 3.1} \]

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with \( \partial \Omega \) of class \( C^2 \) and let \( \alpha \) be the optimal solution corresponding to the prescribed boundary homeomorphism \( g \in H^{3/2}(\partial \Omega) \) of \( \partial \Omega \) on to the boundary \( \partial \Omega_1 \) of a convex set \( \Omega_1 \). Then

\[
J_\alpha(x) = \text{Det}\{\nabla \alpha\} \neq 0, \quad \forall x \in \Omega.
\]

Proof of this result involves complex variable methods and can be found in [8]. We remark here that the proof in [8] requires that the conjugate vector \( \psi \in C(\Omega) \) and this is satisfied here since \( H^2(\Omega) \subset C(\Omega) \) for two dimensional domains.

4 HARMONIC GRIDS IN THREE AND HIGHER DIMENSIONAL DOMAINS

As noted in section (3), the condition (II) of theorem (2.1) is verified for \( \Omega \subset \mathbb{R}^n, n \geq 2 \) if \( g \in H^{(n+1)/2}(\partial \Omega) \). The lemma (3.1) holds for \( \Omega \subset \mathbb{R}^n, n \geq 2 \) and this verifies condition (I) of theorem (2.1). Hence, we need an analogue of proposition (3.1) for \( n \geq 3 \) to verify condition (III) and this is not available.

\[ \text{Open Problem 1} \]

Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \), be a bounded open set with \( \partial \Omega \) of class \( C^{n/2+3} \) and let \( \Omega_1 \subset \mathbb{R}^n \) be a convex bounded domain. Suppose that \( \alpha \) be the optimal solution corresponding to the prescribed boundary homeomorphism \( g \in H^{(n+1)/2}(\partial \Omega) \) of \( \partial \Omega \) on to \( \partial \Omega_1 \). Show that

\[
J_\alpha(x) = \text{Det}\{\nabla \alpha\} \neq 0, \quad \forall x \in \Omega
\]
5 A REMARK ON THE VARIATIONAL FORMULATION

It is possible to generalize the variational grid generation method derived from problem (2.1) as

**Problem 5.1** Find \( \alpha : \Omega \rightarrow \mathbb{R}^n \) such that \( \alpha \in H^1(\Omega) \) and

\[
\mathcal{J}(\alpha) = \frac{1}{2} \| \Lambda \alpha \|_{L^2(\Omega)}^2 + \delta(\alpha, A) + \int_{\Omega} \beta(J_\alpha(x)) dx \rightarrow \inf
\]

with \( \beta(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) convex.

Such functionals also arise in the mathematical theory of nonlinear elasticity [1] (see also [5]). Analysis of such grid generation methods would provide useful informations for computational scientist.

References


