THREE-DIMENSIONAL OVERTURNED TRAVELING WATER WAVES

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Abstract. Traveling gravity-capillary water waves on the interface of a three-dimensional fluid of infinite depth are computed. The vortex sheet formulation with the small scale approximation is used as the mathematical model for the fluid motion. The fluid interface is parameterized isothermally. The traveling wave ansatz for parameterized surfaces is described. Waves are computed using with Fourier collocation and quasi-Newton iteration; large amplitude overturned traveling waves are computed via a dimension-breaking based numerical continuation method.

Key words. overturning, traveling waves, gravity-capillary, dimension-breaking

subject classifications. 35B35, 76B15

1. Introduction

We study periodic waves of the interface between two constant-density fluids undergoing irrotational motions. The fluid regions are infinitely deep in the vertical direction and periodic in the horizontal direction. We seek traveling wave solutions, in which the free surface is of permanent form and steadily translating. This study is fundamentally concerned with waves on a two-dimensional interface, between three-dimensional fluids, which may have overhanging crests (or troughs).

It is the understanding of the authors that no study has been conducted for fully three-dimensional waves which are both overturned and traveling. A number of studies have considered overturning in the time dependent problem, for example [1, 2, 3, 4, 5, 6, 7] with a review in [8]. There are also numerous computations of permanent three-dimensional waves (both traveling and standing waves) in which the interface is parameterized by the horizontal coordinates, for example [9, 10, 11, 12, 13, 14].

The reasons for the absence of previous work on three-dimensional overturned traveling waves are two-fold. First, one must have a three-dimensional formulation of the problem which allows for traveling waves which are overturning. Conformal mappings are by far the most popular technique for the two-dimensional problem, but do not generalize to three-dimensions. In a recent work, the first author and collaborators have developed a formulation which extends to three-dimensions and allows for the computation of traveling waves on interfaces with arbitrary parameterizations [15]. It is in this formulation that this paper proceeds to three-dimensions. The need for such parametric formulations of the water wave problem is not unknown. Alternatively to the track taken here, Bridges and Dias proposed a Hamiltonian formulation which allows for arbitrary interface parameterizations [16].

The second reason for the lack of computations of overhanging three-dimensional traveling waves is the extreme expense of the computation itself, as will be discussed explicitly here, and is reviewed in [8]. In this work, the extreme cost will be partially ameliorated via the use of an approximate model, called the small-scale approximation, proposed in [17] and later used in [18]. The approximation allows the most costly part of the computation, the evaluation of the Birkhoff-Rott integral, to be computed via fast Fourier transforms. The small scale approximation, although exact

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in the small-amplitude limit, is not based on a small-amplitude assumption, and will be used here to compute large amplitude three-dimensional traveling waves, including those with overturned crests and troughs.

For two-dimensional fluids, a significant amount of work has been done in the study of overturned waves both dynamic and steady. We will not try to review them all here. Most relevant to this work are the exact traveling solutions of Crapper [19] and the numerically computed waves of Meiron and Saffman [20], as these two waves are qualitatively similar to the cross-sections of the three-dimensional waves computed here. This paper also is an outgrowth of a number of recent two-dimensional studies by one of the authors. The traveling wave ansatz developed in [15] has since been used extensively to compute two-dimensional overturning traveling waves [21, 22].

The remainder of the paper is organized as follows. In section 2, we present the vortex sheet formulation of the potential flow equations, the small scale approximation to the Birkhoff-Rott equations, and the traveling wave ansatz. These three ingredients combine to give the system of equations which are solved for three-dimensional traveling waves. In section 3, we present the numerical procedure used to compute traveling waves as well as the numerical results. This section includes an example of an overturned three-dimensional traveling wave and discussion of the dimension-breaking continuation procedure used to compute three-dimensional waves. In section 4, we summarize our results and present future research avenues.

2. Formulation

In this work we compute three-dimensional traveling waves in a model for the interface between two-fluids. In particular, we are interested in the case where the fluid interface is overturned, that is, where the vertical displacement is not a function of horizontal cartesian coordinates. To compute such three-dimensional overturning waves, we will represent the interface as a parameterized surface \( \mathbf{X}(\alpha, \beta, t) = (x_1(\alpha, \beta, t), x_2(\alpha, \beta, t), x_3(\alpha, \beta, t)) \). Following Ambrose, Siegel and Tlupova, [1], we will enforce that this parameterization is isothermal, i.e. that

\[
\mathbf{X}_\alpha \cdot \mathbf{X}_\beta = 0, \quad \text{and} \quad G \equiv \|\mathbf{X}_\alpha\|^2 = \lambda \|\mathbf{X}_\beta\|^2 \equiv \lambda E. \tag{2.1}
\]
with

$$\lambda = \frac{\iint G \, d\alpha d\beta}{\iint E \, d\alpha d\beta}$$

We will think of \(\lambda\) as a constant specified at the beginning, describing the aspect ratio of the parameterization (or how much longer the wave is in \(\alpha\) than in \(\beta\)). We choose to set \(\lambda = 1\), and set the ranges for \(\alpha\) and \(\beta\) to be equal to the period of the wave in the corresponding horizontal coordinates, that is

$$x_1(\alpha + P_1, \beta) = x_1(\alpha, \beta) + P_1, \quad \text{and} \quad x_2(\alpha, \beta + P_2) = x_2(\alpha, \beta) + P_2.$$  

Numerically we will take \(x_1 = \alpha + \tilde{x}_1\) and \(x_2 = \beta + \tilde{x}_2\) where the \(\tilde{x}_j\) are periodic corrections, which are more amenable to our Fourier-collocation based numerical method.

Useful in this parameterization are the second fundamental forms

$$L = \hat{X}_{\alpha,\alpha} \cdot \hat{n}, \quad \text{and} \quad N = \hat{X}_{\beta,\beta} \cdot \hat{n}.$$  

In terms of which the mean curvature can be expressed as

$$\kappa = \frac{L\lambda + N}{2\lambda E}.$$  

The isothermal parameterization is the three-dimensional analogy to the arclength parameterization used in [15, 21]. The fluid velocity, \(\vec{W}\), is given in terms of a Birkhoff-Rott integral,

$$\vec{W}(\vec{X}) = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \text{P.V.} \iint (\mu'_\alpha \hat{X}'_{\beta} - \mu'_\beta \hat{X}'_{\alpha}) \times \frac{(\vec{X} - \vec{X}' - nP_1 e_1 - mP_2 e_2)}{\left| \vec{X} - \vec{X}' - nP_1 e_1 - mP_2 e_2 \right|^3} \, d\alpha' \, d\beta'. \quad (2.2)$$

in which the all the primed quantities are evaluated at \((\alpha', \beta')\). The parameter \(P_1\) is the period of the wave in the first horizontal coordinate, \(x_1\), and \(P_2\) is the period of the wave in the second horizontal coordinate \(x_2\). This integral is notoriously difficult to simulate, see [1], and will here be replaced by the small scale approximation of [17, 18]. This approximation takes

$$W \approx \frac{1}{2} H_\alpha \left[ \frac{\mu_\alpha X_\beta \times X_\alpha}{E^2} \right] - \frac{1}{2} H_\beta \left[ \frac{\mu_\beta X_\alpha \times X_\beta}{E^2} \right]$$

which captures the near-singular behavior of the integral and avoids the significant difficulties associated with computing the Birkhoff-Rott integral, see for example Beale’s discussion of the convergence of this integral [23]. The operators \(H_\alpha\) and \(H_\beta\) are the Riesz transforms. The Riesz transforms are diagonalized by the Fourier transform, and have multiplicative Fourier symbols,

$$\hat{H}_\alpha f = -i \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \hat{f}, \quad \text{and} \quad \hat{H}_\beta f = -i \frac{k_2}{\sqrt{k_1^2 + k_2^2}} \hat{f}.$$  

A study of the overturning waves with the full Birkhoff-Rott integral using the algorithm of Siegel and colleagues [1] is being pursued separately.
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The Bernoulli equation for the evolution of the vortex sheet strength is

\[ \mu_t = \tau \kappa + \left( \frac{\mu_\alpha}{\sqrt{E}} (V_1 - W \hat{t}_1) + \frac{\mu_\beta}{\sqrt{E}} (V_2 - W \hat{t}_2) \right) + At \left( |W|^2 + 2W \cdot \hat{t}_1 (V_1 - W \hat{t}_1) + 2W \cdot \hat{t}_2 (V_2 - W \hat{t}_2) - \frac{\mu_\alpha^2 + \mu_\beta^2}{4E} - gx_3 \right) \]  
(2.3)

Here \( V_j \) are the tangential components of the velocity of interface in our parameterization, not to be confused with \( W \cdot \hat{t}_j \), the velocity of fluid particles on the interface. The parameter \( \tau \) is the surface tension coefficient, \( g \) is gravity, and \( At = \rho_1 - \rho_2 / \rho_1 + \rho_2 \) is the Atwood ratio, comparing the densities of the upper, \( \rho_2 \), and lower, \( \rho_1 \), fluids.

The kinematic equation for the interface is

\[ \dot{X}_i = U \hat{n} + V_1 \hat{t}_1 + V_2 \hat{t}_2 \]  
(2.4)

where \( U \) is the physical normal velocity to the interface and \( V_j \) are chosen to preserve the isothermal parameterization. For a general interface motion, to preserve an isothermal parameterization requires that the \( V_j \) solve an elliptic equation, as discussed in [1]. For steadily translating interfaces, as is the case for traveling waves, the \( V_j \) can be determined by the kinematic condition (2.4) coupled with the prescription that the interface is traveling in the \( x_1 \)-direction, \( \dot{X}_i = (c,0,0) \), yielding

\[ \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{n} & \hat{t}_1 & \hat{t}_2 \end{pmatrix} \begin{pmatrix} U \\ V_1 \\ V_2 \end{pmatrix}. \]

If one considers the speed \( c \), and interface location \( X \) to be known, then \( U \) and \( V_j \) are specified

\[ V_j = c(\hat{t}_j)_1, \quad \text{and} \quad U = c(\hat{n})_1. \]  
(2.5)

From the perspective of the kinematic equation, any interface shape is allowable, so long as the velocity of the interface is coupled to the shape by (2.5). If one chooses, as we will here, to parameterize in a frame moving with the wave, then the interface shape is independent of time. A sufficient condition for traveling waves in such a frame is that the vortex sheet strength also has \( \mu_t = 0 \). The coupling of \( \mu_t = 0 \) and (2.5), play the role of the traveling wave ansatz in this formulation. Combining this ansatz with equation (2.3) under an isothermal parameterization gives the equations for traveling waves.

To compute a traveling wave requires finding four functions \( x_1, x_2, x_3, \) and \( \mu \) as well as a speed \( c \), which solve four equations,

\[ 0 = \tau \kappa + \frac{1}{\sqrt{E}} (\tilde{V} \cdot \nabla) \mu + At \left( |W|^2 + 2W \cdot \hat{t}_1 \tilde{V}_1 + 2W \cdot \hat{t}_2 \tilde{V}_2 - \frac{1}{4E} |\nabla \mu|^2 - gx_3 \right), \]  
(2.6a)

\[ 0 = c(\hat{n})_1 - W \cdot \hat{n}, \]  
(2.6b)

\[ 0 = X_\alpha \cdot X_\beta, \]  
(2.6c)

\[ 0 = G - \lambda E, \]  
(2.6d)

in which \( \tilde{V}_j = c(\hat{t}_j)_1 - W \cdot \hat{t}_j \), and \( \tilde{V} = (\tilde{V}_1, \tilde{V}_2) \). We append another equation fixing the wave amplitude to close the system. The measure of wave amplitude will vary in our numerical method. For the results in the following section we alternately use the crest height, total displacement, and the amplitude of a Fourier mode of the third coordinate.
3. Results and Discussion

The numerical method used is a combination of Fourier collocation and a quasi-Newton iteration, similar to those used in [15, 24, 25]. The unknown functions $x_1, x_2, x_3,$ and $\mu$ are all real functions of two parametric variables $\alpha$ and $\beta$. We compute both planar waves, which are constant in the direction transverse to propagation, and fully three-dimensional waves, which depend non-trivially on both $\alpha$ and $\beta$.

To compute overturned fully three-dimensional waves, we use a dimension-breaking approach. First a branch of planar waves are computed. Because these waves do not depend on $\beta$, one needs only to compute the profile at a single location. This dimension reduction makes computing large amplitude planar waves significantly less expensive than computing fully three-dimensional branches of traveling waves. Next, branches of waves with non-trivial $\beta$ dependence are computed as bifurcations from the branch of planar waves. These waves have transverse ($\beta$ direction) periodicity which depends on the amplitude from which they bifurcate. The numerical cost to compute such waves is significantly greater than the planar waves, however since one can pay the planar cost to reach large amplitude, we are able to compute well resolved, large amplitude, overturned, fully three-dimensional waves. An example of the dimension breaking bifurcation is depicted in figure 3.1.

The fully three-dimensional Fourier collocation begins with $M_a$ points in $\alpha$ and $M_b$ points in $\beta$, where $(\alpha, \beta) \in [-\pi, \pi) \times [-\pi, \pi)$. Thus direct projection of these functions onto Fourier modes would yield $4M_a M_b$ unknowns. Problem symmetries allow for this number to be reduced significantly.

The number of unknowns are reduced via the following sequence of arguments. The solutions sought are real functions, therefore one needs only compute the Fourier coefficients in one quadrant of Fourier space. Second, symmetries allow for one to look for $x_3(\alpha, \beta)$ which is doubly even (in both $\alpha$ and $\beta$) and $\mu(\alpha, \beta)$ which is odd in $\alpha$ and even in $\beta$. Similarly we seek $x_1(\alpha, \beta)$ which is odd in $\beta$ and even in $\alpha$, and
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$x_2(\alpha,\beta)$ which is even in $\alpha$ and odd in $\beta$. The last two are simply choices of how the parametric variables $\alpha$ and $\beta$ are aligned with respect to the horizontal coordinates $x_1,x_2$. These parity choices allow one to compute Fourier coefficients which are either pure real (if the function is even in both variables) or pure imaginary (if the function is even in one variable and odd in the other). Many of the spatial averages of these functions need not be solved for; odd functions have zero spatial means. The end result is that to compute a traveling wave one must solve for $M_a M_b - M_b - \frac{1}{2} M_a + 1$ Fourier coefficients (as well as the speed $c$).

The system of equations we solve are the projection of equations (2.6) into Fourier space. For $(x,\mu)$ with the above described parity, the equations support similar symmetries and give the same number of non-trivial Fourier coefficients as the equations. We couple to this system an equation specifying the size of the traveling wave to close the system. This last equation is used as our continuation parameter, the choice of which will vary along a branch of traveling waves. For small amplitude we use the total displacement $h = \max(x_3) - \min(x_3)$. For large amplitude we sometimes observe turning points, where the branch has a locally maximal displacement, in which case we switch to another measure of the wave size, for example a Fourier coefficient of $x_3$.

The primary numerical cost in computing traveling waves is the filling (and storing) of the approximation of the Jacobian in the quasi-Newton solver. We ameliorate this to some extent via Broyden’s update, and by re-using Jacobian’s during the continuation procedure. This does not avoid the expense of storing large Jacobian matrices. The highest resolution wave computed had $M_a = 128$ and $M_b = 256$; at this resolution the Jacobian is $32450 \times 32450$, which in IEEE type double costs 8.4GB just to store. Rather than push the computational limits of our machine, we have chosen to present only waves which are very regular. This allows for highly resolved computations at a relatively small number of points (see figure 3.2, which has $M_a = 128, M_b = 32$).

For this first work on overturned traveling waves, the small-scale approximation to the Birkhoff-Rott integral is employed. This reduces the cost of computing $W$ to $O(M_a M_b \log(M_a M_b))$. Simulation of the full Birkhoff-Rott integral is possible, but significantly more complicated, and is being pursued separately. The difficulty of simulating the Birkhoff-Rott integral for three-dimensional fluids is well known [11]. A modern fast technique is that of Siegel and colleagues [1] which combines Ewald summation, matched near-field and far field expansions and a tree code, in the spirit of the “Fast Multipole Method” [5, 26].

To compute a traveling solution with a quasi-Newton iteration requires an initial guess. For small amplitude, it is natural to use a Stokes’ expansion to compute one traveling wave, and then compute larger amplitude waves via numerical continuation. This approach is numerically costly, as one must pay the cost of computing a fully three-dimensional wave at every amplitude along the branch of traveling waves. We take an alternate tactic, following dimension breaking bifurcations from a planar (two-dimensional) traveling wave.

To compute three-dimensional bifurcations from a planar wave, our computations begin with a planar solution

\[ x_1 = \tilde{x}(\alpha), \quad x_2 = \beta, \quad x_3 = \tilde{z}(\alpha), \quad \text{and} \quad \mu = \tilde{\mu}(\alpha) \]  

(3.1)

and perturb the displacement and vortex sheet strength with small spatial variation of period $L$. We use initial guesses with $x_1$ and $x_2$ identical to the planar wave (as
above) and perturb in the displacement and vortex sheet strength

\[ x_3 = \tilde{z}(\alpha) \left( 1 + \delta \cos \left( \frac{2\pi}{L} \beta \right) \right), \quad \text{and} \quad \mu = \tilde{\mu}(\alpha) \left( 1 + \delta \cos \left( \frac{2\pi}{L} \beta \right) \right). \]  

(3.2)

We observe transversely periodic branches of traveling waves exist for a countable set of \( L \) for each amplitude \( h \). We interpret this observation as the existence of branches of waves of a single periodicity, \( L \), which are of course also periodic on periods \( mL \) for \( m \in \mathbb{N} \). We compute only the case when \( m = 1 \), reporting waves which are periodic exactly once in the transverse domain. After this restriction, there appears to be only one period for which three-dimensional traveling waves bifurcate for each amplitude of a branch of planar traveling waves.

A significant challenge in this procedure is guessing the period at which the bifurcations occur. We compute these by guessing a large period at small amplitude, where the wave profiles at different periods are similar, thus they serve as a good initial guesses for nearby periods. Next, we increase amplitude and continue in the period of our initial guess, always using the ansatz (3.2) for the displacement and vortex sheet strength. This guess is ad-hoc, not based on formal asymptotics. We would prefer to have an explicit formula for waves which are weakly varying in the transverse direction, as in [27], however we believe such a solution is impossible to find; it would require solving a linear non-constant coefficient PDE whose coefficients are only known numerically. The absence of such an asymptotic formula results in regions of parameter space where our initial guess (3.2) is not good enough to compute the dimension breaking bifurcations. That being said, we were able to compute dimension breaking along the entire branch of planar traveling waves in many configurations. One such configuration is depicted in figure 3.1, in which we compute dimension breaking bifurcations from a planar wave with \( \sigma = 1/8 \).

Generally we computed only small departures from the planar traveling waves, i.e., the local bifurcation structure. We did observe a case where the global bifurcation was also a small departure from planar. In the left panel of figure 3.1, observe the second and third to largest reported dimension breaking bifurcations merge. These two bifurcations thus form the “return to trivial” global bifurcation. This phenomenon occurs in the two-dimensional setting as well, and is described in case (e) of the global bifurcation theorem of [28] and was numerically computed in [22].

In one space dimension there are two qualitatively different types of overturned traveling water waves. The first resemble the Crapper’s wave, see the left panel of figure 1.1, which limits on an enclosed bubble and has large curvature in the neighborhood of this bubble. The second resembles the waves computed by Meiron and Saffman [20] and are more regular, see the right panel of figure 1.1. Numerically, the latter requires many fewer points to resolve; the planar wave has Fourier modes decaying to machine precision right at wave number \( k = 32 \) (corresponding to \( M_a = 64 \)). In this work, we compute dimension breaking bifurcations near planar waves at two Bond numbers \( \sigma = \frac{k^2}{\tau^2} \), here \( k \) is the typical wave number of the planar wave based on its longitudinal period. The planar waves with \( \sigma = 1/8 \) resemble that of Crapper. The planar waves with \( \sigma = -1/10 \) resemble the waves of Meiron and Saffman.

An example of an overturned three-dimensional traveling wave is depicted in figure 3.2. This wave was computed via dimension breaking numerical continuation from a planar wave \( \sigma = -1/10 \). In this configuration the overturned planar wave very regular (see the right panel of figure 1.1). Three-dimensional traveling waves were also computed via dimension breaking numerical continuation from planar waves.
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Fig. 3.2. An overturned fully three-dimensional traveling wave solution to (2.6), using the small-scale approximation to the Birkhoff-Rott integral. This wave was computed with $At = 1$, $\tau = 2, g = -0.2$, with $Ma = 128$ and $Mp = 32$. In the left panel is the three-dimensional surface, in the right panel is a slice at $y = -\pi/2$. In the right panel, the grid-points are marked with solid circles.

with $\sigma = 1/8$. In the latter configuration we were computed a number of dimension-breaking bifurcations, see figure 3.1. The waves computed in this figure all have interfaces which are functions of the horizontal cartesian coordinates. In this case the overturned waves requires significantly more points to resolve, and we were only able to resolve waves whose interfaces did not overturn.

Although we believe that there are fully three-dimensional overturned traveling waves at generic Bond numbers, bifurcations from waves with narrow bubbles (i.e. Crapper-like) require too many points for our current capabilities. Numerically we see evidence that overturned traveling waves exist in this setting, but we are only able to compute their under-resolved approximations. We are currently pursuing a study of overturned three-dimensional traveling waves in this more expensive case, as well as computing full Birkhoff-Rott integral, rather than its small scale approximation, using the Air Force Research Laboratory’s supercomputing resource center [29].

4. Conclusion Fully three-dimensional overhanging traveling waves are computed in the vortex sheet equations for water waves with surface tension. The small-scale approximation is used in the Birkhoff-Rott integral for the velocity field. A traveling wave ansatz for parameterized surfaces is presented. Large amplitude overhanging waves are computed via dimension-breaking continuation from planar traveling waves. Future research directions include computing these waves with the full Birkhoff-Rott integral, instead of the small scale approximation. Also desirable would be parameter space explorations, computing overturned traveling waves with fine structure, as would be the case for three-dimensional bifurcations from Crapper’s wave. It is also natural to question as to whether any of these waves are stable to perturbations, and should they be unstable, how do instabilities manifest in the time-dependent problem.

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