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Short communication

A counterexample to Beder's conjectures about Hadamard matrices

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ABSTRACT

In this note we provide a counterexample which resolves conjectures about Hadamard matrices made in this journal. Beder [1998. Conjectures about Hadamard matrices. Journal of Statistical Planning and Inference 72, 7–14] conjectured that if $\boldsymbol{H}$ is a maximal $m \times n$ row-Hadamard matrix then m is a multiple of 4; and that if n is a power of 2 then every row-Hadamard matrix can be extended to a Hadamard matrix. Using binary integer programming we obtain a maximal 13×32 row-Hadamard matrix, which disproves both conjectures. Additionally for n being a multiple of 4 up to 64, we tabulate values of m for which we have found a maximal row-Hadamard matrix. Based on the tabulated results we conjecture that a $m \times n$ row-Hadamard matrix with $m \geqslant n-7$ can be extended to a Hadamard matrix.

1. Introduction

This note resolves two conjectures on Hadamard matrices made by Beder (1998) in this journal. A square matrix with entries of ± 1 is said to be *Hadamard* if its rows and columns are orthogonal. A basic property of Hadamard matrices is that $n = 2$ or n is a multiple of 4. An $m \times n$ matrix with entries of ± 1 is *row-Hadamard* if its rows are orthogonal, and a row-Hadamard matrix is *maximal* if it cannot be extended by adding rows. Beder's conjectures dealt with values of n and m for which a maximal row-Hadamard matrix exists.

The following result from Vijayan (1976) provides initial restrictions on the possible values of n and m:

Theorem 1. *Let n be a multiple of 4. If $\boldsymbol{H}$ is an $m \times n$ row-Hadamard matrix with $m \geqslant n-4$ then $\boldsymbol{H}$ can be extended to a Hadamard matrix of order n.*

Beder (1998) gave some rudimentary results on values of m and n for which a maximal row-Hadamard matrix can be constructed: If n is odd then $m = 1$; if $n = 2 \pmod 4$ then $m = 2$; $n = 4$ implies $m = 4$. Furthermore he showed the existence of a maximal row-Hadamard matrix with $m = 4$ when n is an odd multiple of 4. Then he used an exhaustive construction to identify maximal row-Hadamard matrices with n columns and tabulated the values of m for which maximal row-Hadamard matrices were found. The values of n and m that he identified led to the following two conjectures:

Conjecture 1. *Let n be a multiple of 4 and let $\boldsymbol{H}$ be a maximal $m \times n$ row-Hadamard matrix. Then m is a multiple of 4.*

Conjecture 2. *If n is a power of 2, then every row-Hadamard matrix can be extended to a Hadamard matrix.*

We employed a binary integer programming technique which improved upon the exhaustive search used by Beder. Row-Hadamard matrices were shown to be maximal when integer programming indicated that an extension was infeasible. All

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Table 1
Values of n and m for which maximal row-Hadamard matrices were found.

n	m
8	8
12	4, 12
16	16
20	4, 12, 20
24	8, 12, 16, 24
28	4, 8, 10, 11, 12, 13, 14, 16, 20, 28
32	8, 12, 13, 14, 15, 16, 17, 18, 20, 24, 32
36	4, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 28, 36
40	8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 32, 40
44	4, 8, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24
48	8, 14, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 36, 40, 48
52	4, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25
56	8, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, *28, 32, 40, 56*
60	4, 8, 12, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27
64	20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, *64*

Values of m in italics arose as a consequence of Lemma 1 rather than construction. Results for $n \leqslant 20$ are complete.

integer programs were solved using the solver Cplex 10.1. Using this integer programming method we identified a maximal row-Hadamard matrix with $n=32$ and $m=13$ rows, whose existence answers both conjectures in the negative.

In Section 2 we present the integer programming-based algorithm. The results produced by using this algorithm are presented in Section 3, including the 13×32 matrix which resolves both conjectures. For values $8 \leqslant n \leqslant 64$ which are multiples of 4, we tabulate values of m for which maximal row-Hadamard matrices were found. In Section 3 we also present a lemma that describes a method of constructing a $n_1 n \times n_1 n_2$ maximal row-Hadamard matrix from a $n_1 \times n_1$ Hadamard matrix and an $m \times n_2$ maximal row-Hadamard matrix.

2. The binary integer programming algorithm

The following algorithm was used to find maximal row-Hadamard matrices.

1. Begin with a $1 \times n$ row vector $(1, \ldots, 1)$. This is the first row of $\boldsymbol{H}$.
2. Introduce variables $x_{21}, \ldots, x_{2n} \in \{0,1\}$ with the constraint $\sum_{i=1}^{n}(2x_{2i}-1)=0$. Then for randomly chosen binary constants $c_1, \ldots, c_n$ selected with equal probability, maximize the objective function $\sum_{i=1}^{n} c_i x_{2i}$. For any solution $x_{21}, \ldots, x_{2n}$ set the second row of $\boldsymbol{H}$ to $2x_{21}-1, \ldots, 2x_{2n}-1$.
3. For $k \geqslant 3$ introduce variables $x_{k1}, \ldots, x_{kn} \in \{0,1\}$ with constraints
   (i) $\sum_{i=1}^{n}(2x_{ki}-1)=0$, and
   (ii) $\sum_{i=1}^{n}(2x_{ki}-1)(2x_{2i}-1)=0, \ldots, \sum_{i=1}^{n}(2x_{ki}-1)(2x_{k-1i}-1)=0$.
   Using the same binary constants $c_1, \ldots, c_n$ from step 2, maximize the objective function $\sum_{i=1}^{n} c_i x_{ki}$. For any solution $x_{k1}, \ldots, x_{kn}$ set the kth row of $\boldsymbol{H}$ to $[2x_{k1}-1, \ldots, 2x_{kn}-1]$.

The algorithm continues until the constraints become infeasible. At each step the constraints force $\boldsymbol{H}$ to be row-Hadamard, and when the constraints become infeasible $\boldsymbol{H}$ is maximal row-Hadamard.

Note that Constraint (i) gives $\sum_{i=1}^{n} x_{ki} = n/2$ and Constraint (ii) yields

$$2\sum_{i=1}^{n}(2x_{ji}-1)x_{ki} = \sum_{i=1}^{n}(2x_{ji}-1)$$

for $j=2, \ldots, k-1$. But $\sum_{i=1}^{n}(2x_{ji}-1)=0$ from Constraint (i) at earlier iterations. Then at the kth iteration, Constraints (i) and (ii) can be expressed as $\boldsymbol{Hx}=\boldsymbol{b}$ where $\boldsymbol{H}$ is the row Hadamard matrix obtained in the previous iteration, $\boldsymbol{x}=(x_{k1}, \ldots, x_{kn})^T$ and $\boldsymbol{b}=(n/2, 0, \ldots, 0)^T$.

3. Results

For $n=8$ using Theorem 1 it is easy to see that there are no maximal row-Hadamard matrices for $m<n$. For $n=12,16,20$ all maximal row-Hadamard matrices were located by Sun et al. (2004) and verified by Bulutoglu and Ryan (2009). For $n=24,28,32,36,40,44$ and 48 we ran the algorithm in Section 2 100,000 times. For $n=52,56,60$ and 64 we ran the algorithm 1000 times as these cases were much slower to run than the other cases studied. In Table 1, we show the values of m for which we obtained maximal row-Hadamard matrices. In particular, the case $n=32$ and $m=13$ resolves Beder's conjectures, so in Table 2 we show a maximal row-Hadamard matrix with these parameters.

Table 1 suggests that the stronger result obtained by replacing $n-4$ with $n-7$ in Theorem 1 might be true. Based upon this observation we state the following conjecture:

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Table 2
A Maximal row-Hadamard matrix with $n = 32$, $m = 13$.

Row	
1	++++++++++++++++++++++++++++++++
2	−+−++−++−+++++−−−−+−++−−−−+−−−++
3	+++−−−++−+++−−−−−+−++−++−−+++−−−
4	−−−−−+−−++++++++−−−−+−++−+++−−+−
5	++−+−+++−−−−++++−++−−−++−−−+−−−+
6	+−+++−−−++−−+−−−−−++++++−+−+−−−+
7	−+−−+−+++−−−−+−−+−−−+−++−+−+++++
8	−−−+−+−+−+−+−−−−+++−+++++−−+−++−
9	−−+++−+−−−+−−++−+−++−−+++−++−−+−
10	++−−+−−−−−+++−−++−−−−++++−++−+−+
11	−++−−++−+++−−−−+−−+−−++−+−−++−++
12	−+−+++−+−+−−−+−+−−−+−+−++++++−−−
13	−−+−++++−−−++−+−−−+−−++−−+++++−−

Conjecture 3. *Let n be a multiple of* 4. *If* $\boldsymbol{H}$ *is an* $m \times n$ *row-Hadamard matrix with* $m \geqslant n - 7$ *then* $\boldsymbol{H}$ *can be extended to a Hadamard matrix of order n.*

A maximal row-Hadamard matrix can be used to obtain an infinite class of maximal row-Hadamard matrices. It can also be used to construct a maximal row-Hadamard matrix corresponding to some of the parameter combinations in Table 1. This is summarized in the following lemma. The proof is omitted for brevity.

Lemma 1. *Let* $\boldsymbol{H}_0$ *be an* $n_1 \times n_1$ *Hadamard matrix. Also let* $\boldsymbol{H}_1$ *be an* $m \times n_2$ *maximal row-Hadamard matrix. Then* $\boldsymbol{H}_0 \otimes \boldsymbol{H}_1$ *is an* $n_1 m \times n_1 n_2$ *maximalrow-Hadamard matrix.*

As a special case, we obtain a generalization of Lemma 2.1 in Beder (1998). For t odd, when $\boldsymbol{H}_1$ is a $1 \times t$ row vector of all 1s, this result produces a $4\alpha \times 4\alpha t$ maximal row-Hadamard matrix.

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References

Beder, Jay H., 1998. Conjectures about Hadamard matrices. J. Statist. Plann. Inference 72, 7–14.
Bulutoglu, D., Ryan, K.J., 2009. A step down algorithm with integer programming for classifying *s*-level orthogonal arrays, in preparation.
Sun, D.X., Li, W., Ye, K.Q., 2004. An algorithm for sequentially constructing non-isomorphic orthogonal designs and its applications. SUNYSB Preprint, AMS-02-13.
Vijayan, K., 1976. Hadamard matrices and submatrices. J. Aust. Math. Soc. Ser. A 22, 469–475.

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