Appl Math Optim 46:31–53 (2002)
DOI: 10.1007/s00245-002-0734-6

Applied Mathematics & Optimization

Stochastic 2-D Navier–Stokes Equation*

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Abstract. In this paper we prove the existence and uniqueness of strong solutions for the stochastic Navier–Stokes equation in bounded and unbounded domains. These solutions are stochastic analogs of the classical Lions–Prodi solutions to the deterministic Navier–Stokes equation. Local monotonicity of the nonlinearity is exploited to obtain the solutions in a given probability space and this significantly improves the earlier techniques for obtaining strong solutions, which depended on pathwise solutions to the Navier–Stokes martingale problem where the probability space is also obtained as a part of the solution.

Key Words. Stochastic Navier–Stokes equation, Maximal monotone operator, Markov–Feller semigroup, Stochastic differential equations.

AMS Classification. 35Q30, 76D05, 60H15.

1. Introduction

The mathematical theory of the Navier–Stokes equation is of fundamental importance to a deep understanding, prediction and control of turbulence in nature and in technological applications such as combustion dynamics and manufacturing processes. The incompressible Navier–Stokes equation is a well accepted model for atmospheric and ocean dynamics. The stochastic Navier–Stokes equation has a long history (e.g., see [6] and [17] for two of the earlier studies) as a model to understand external random

* The research by S. S. Sritharan was supported by the ONR Probability and Statistics Program.

forces. In aeronautical applications random forcing of the Navier–Stokes equation models structural vibrations and, in atmospheric dynamics, unknown external forces such as sun heating and industrial pollution can be represented as random forces. In addition to the above reasons there is a mathematical reason for studying stochastic Navier–Stokes equations. It is well known that the invariant measure of the Navier–Stokes equation is not unique. A well known conjecture of Kolmogorov suggests that addition of noise would reduce the number of physically meaningful invariant measures.

A rigorous theory of the stochastic Navier–Stokes equation has been a subject of several papers. Several approaches have been proposed, from the classic paper by Bensoussan and Temam [4] to some more recent results, e.g., by Bensoussan [3], by Flandoli and Gatarek [10] and by Sritharan [20]. The reader is referred to the books by Vishik and Fursikov [22] and Capinski and Cutland [5] for a comprehensive treatment. Most papers rely on martingale-type methods and a direct theory of strong solutions providing the stochastic analog of the well known Lions and Prodi [14] solvability theorem for the deterministic Navier–Stokes equation remained open in the past. In this paper we prove exactly such a result exploiting a local monotonicity property. Our method covers both bounded and unbounded domains since it does not rely on compactness methods. The results of this paper have been very useful in treating impulse and stopping time problems (see [15]) and also show promise in obtaining local (stochastic) strong solutions to 3-D bounded and unbounded domains, which is currently an open problem.

In the rest of this section we formulate the abstract Navier–Stokes problem. Throughout the paper we consider the case of bounded domains to enhance readability and indicate the appropriate modifications for unbounded domains. In Section 2 we describe the local monotonicity property of the Navier–Stokes operators. The required interpolation theorems (all valid for arbitrary unbounded domains) are provided for completeness. We then establish certain new a priori estimates involving exponential weight for the deterministic Navier–Stokes equation. In Section 3 we imitate these exponentially weighted estimates for the stochastic case. These estimates play a fundamental role in the proof of the existence and uniqueness of strong solutions proved in the second half of Section 3. The monotonicity argument used here is a generalization of the classical Minty–Browder method for dealing with local monotoniticity. Finally we also prove the Feller property of the stochastic process.

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\mathcal{O}$. Denote by $\mathbf{u}$ and p the velocity and the pressure fields. The Navier–Stokes problem (with Newtonian constitutive relationship) is as follows:

$$\partial_t \mathbf{u} - \nu \triangle \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in} \quad \mathcal{O} \times (0, T), \tag{1.1}$$

with the conditions

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \mathcal{O} \times (0, T), \\ \mathbf{u} = 0 & \text{in} \quad \partial\mathcal{O} \times (0, T), \\ \mathbf{u} = \mathbf{u}_0 & \text{in} \quad \mathcal{O} \times \{0\}, \end{cases} \tag{1.2}$$

where $\mathbf{f}$ is a given forcing field. It is well known (e.g., [7], [13], [12], [21] and [23]) that by means of divergent free Hilbert spaces $\mathbb{H}$, $\mathbb{V}$ (and its dual $\mathbb{V}'$) and the *Helmhotz–Hodge* orthogonal projection $P_{\mathbb{H}}$, the above classical form of the Navier–Stokes equation can be re-written in the following abstract form:

$$\partial_t \mathbf{u} + A\mathbf{u} + B(\mathbf{u}) = \mathbf{f} \qquad \text{in} \quad \mathbb{L}^2(0, T; \mathbb{V}'), \tag{1.3}$$

with the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \mathbb{H}, \tag{1.4}$$

where now $\mathbf{u}_0$ belong to $\mathbb{H}$ and the field $\mathbf{f}$ is in $\mathbb{L}^2(0, T; \mathbb{H})$. The standard spaces used are as follows:

$$\mathbb{V} = \{\mathbf{v} \in \mathbb{H}^1_0(\mathcal{O}, \mathbb{R}^2) \colon \nabla \cdot \mathbf{v} = 0 \text{ a.e. in } \mathcal{O}\}, \tag{1.5}$$

with the norm

$$\|\mathbf{v}\|_{\mathbb{V}} := \left(\int_{\mathcal{O}} |\nabla \mathbf{v}|^2 \, dx \right)^{1/2} = \|\mathbf{v}\|, \tag{1.6}$$

and $\mathbb{H}$ is the closure of $\mathbb{V}$ in the $\mathbb{L}^2$-norm

$$\|\mathbf{v}\|_{\mathbb{H}} := \left(\int_{\mathcal{O}} |\mathbf{v}|^2 \, dx \right)^{1/2} = |\mathbf{v}|. \tag{1.7}$$

The linear operators $P_{\mathbb{H}}$ (Helmhotz–Hodge projection) and A (Stokes operator) are defined by

$$\begin{cases} P_{\mathbb{H}} \colon \mathbb{L}^2(\mathcal{O}, \mathbb{R}^2) \longrightarrow \mathbb{H}, & \text{orthogonal projection,} \qquad \text{and} \\ A \colon \mathbb{H}^2(\mathcal{O}, \mathbb{R}^2) \cap \mathbb{V} \longrightarrow \mathbb{H}, & A\mathbf{u} = -\nu P_{\mathbb{H}} \triangle \mathbf{u}, \end{cases} \tag{1.8}$$

and the nonlinear operator

$$B \colon \mathcal{D}_B \subset \mathbb{H} \times \mathbb{V} \longrightarrow \mathbb{H}, \qquad B(\mathbf{u}, \mathbf{v}) = P_{\mathbb{H}}(\mathbf{u} \cdot \nabla \mathbf{v}), \tag{1.9}$$

with the notation $B(\mathbf{u}) = B(\mathbf{u}, \mathbf{u})$, and, clearly, the domain of B requires that $(\mathbf{u} \cdot \nabla \mathbf{v})$ belongs to the Lebesgue space $\mathbb{L}^2(\mathcal{O}, \mathbb{R}^2)$.

Using the Gelfand triple (duality) $\mathbb{V} \subset \mathbb{H} = \mathbb{H}' \subset \mathbb{V}'$ we may consider A as mapping $\mathbb{V}$ into its dual $\mathbb{V}'$. The inner product in the Hilbert space $\mathbb{H}$ (i.e., $\mathbb{L}^2$-scalar product) is denoted by $(\cdot, \cdot)$ and the induced duality by $\langle \cdot, \cdot \rangle$. It is convenient to notice that for $\mathbf{u} = (u_i)$, $\mathbf{v} = (v_i)$ and $\mathbf{w} = (w_i)$ we have

$$\langle A\mathbf{u}, \mathbf{w} \rangle = \nu \sum_{i,j} \int_{\mathcal{O}} \partial_i u_j \, \partial_i w_j \, dx \tag{1.10}$$

and

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = \sum_{i,j} \int_{\mathcal{O}} u_i \, \partial_i v_j w_j \, dx. \tag{1.11}$$

An integration by part and Hölder inequality yields

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = -\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v} \rangle, \tag{1.12}$$

$$|\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle| \le \sum_{i,j} \|u_i w_j\|_{L^2(\mathcal{O}, \mathbb{R}^2)} \|\partial_i v_j\|_{L^2(\mathcal{O}, \mathbb{R}^2)}, \tag{1.13}$$

and in each term of the right-hand side we can use $\mathbb{L}^4$-norms to estimate the product $u_i v_j$. Notice that in getting equality (1.12) we use the fact that $\mathbf{u}$ is divergent free (i.e., $\nabla \cdot \mathbf{u} = 0$), but $\mathbf{v}$ and $\mathbf{w}$ are not necessarily divergent free. Hence, we have $\langle B(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle = 0$ and $\langle B(\mathbf{u}, \mathbf{v}), \mathbf{v}^3 \rangle = 0$, where $\mathbf{v}^3$ is defined by components, i.e., $\mathbf{v}^3(x, t) := [v_i^3(x, t)]$.

2. Some Estimates

Before setting the stochastic PDE, we give some elementary estimates.

Lemma 2.1. *If φ and ψ are smooth functions with compact support in $\mathbb{R}^2$, then*

$$\|\varphi\psi\|_{L^2}^2 \le 4\|\varphi\,\partial_1\varphi\|_{L^1}\,\|\psi\,\partial_2\psi\|_{L^1}, \tag{2.1}$$

$$\|\varphi\|_{L^4}^4 \le 2\|\varphi\|_{L^2}^2\,\|\nabla\varphi\|_{\mathbb{L}^2}^2. \tag{2.2}$$

Moreover, if $C_{\mathcal{O}}$ denotes the diameter of the domain $\mathcal{O}$, and φ, ψ have support in $\bar{\mathcal{O}}$, then we have

$$\|\varphi\psi\|_{L^2} \le C_{\mathcal{O}}\|\partial_1\varphi\|_{L^2}\,\|\partial_2\psi\|_{L^2}, \tag{2.3}$$

$$\|\varphi\psi\|_{L^2}^2 \le C_{\mathcal{O}}\|\partial_1\varphi\|_{L^2}^2\,\|\psi\,\partial_2\psi\|_{L^1}. \tag{2.4}$$

Clearly, all estimates remain true for functions in $H_0^1(\mathcal{O})$.

Proof. Actually, the result (2.2) is well known. We give a proof only for the sake of completeness. First, use the equality

$$\varphi(x, y) = \int_{-\infty}^{x} \partial_1\varphi(s, y)\,ds = \int_{-\infty}^{y} \partial_2\varphi(x, t)\,dt$$

to obtain

$$\|\varphi\psi\|_{L^1} \le \|\partial_1\varphi\|_{L^1}\,\|\partial_2\psi\|_{L^1},$$

for any φ and ψ. Hence, applying the above estimate for φ^2 and ψ^2 instead of φ and ψ, we get

$$\|\varphi\psi\|_{L^2}^2 \le 4\|\varphi\,\partial_1\varphi\|_{L^1}\,\|\psi\,\partial_2\psi\|_{L^1},$$

which implies the first part. Similarly, starting with

$$|\varphi(x, y)|^2 \le \left|\int_{-\infty}^{x} \partial_1\varphi(s, y)\,ds\right|^2 \le C_{\mathcal{O}} \int_{-\infty}^{+\infty} |\partial_1\varphi(s, y)|^2\,ds$$

we prove the desired estimates. □

Notice that in 3-D, we can use estimate (2.3) to get

$$\int_{\mathbb{R}^2} |\varphi(x, y, z)|^4\,dx\,dy \le 2\left(\int_{R^2} u^2(x, y, z)\,dx\,dy\right)\left(\int_{R^2} |\nabla u|^2(x, y, z)\,dx\,dy\right),$$

where we bound $u^2(x, y, z)$ by $\int_{\mathbb{R}} |u(x, y, z)\,\partial_z u(x, y, z)|\,dz$ to deduce

$$\|\varphi\|_{L^4(\mathbb{R}^3)}^4 \le 4\|\varphi\|_{L^2(\mathbb{R}^3)}\,\|\nabla\varphi\|_{\mathbb{L}^2(\mathbb{R}^3)}^3, \tag{2.5}$$

which is similar to (2.3).

The previous Lemma implies that $\mathbb{H} \cap \mathbb{L}^4(\mathcal{O}, \mathbb{R}^2)$ contains $\mathbb{V}$ as a dense subspace (even in 3-D). Moreover, $L^2(0, T; \mathbb{V}) \cap L^\infty(0, T; \mathbb{H})$ is contained in $\mathbb{L}^4((0, T) \times \mathcal{O}, \mathbb{R}^2)$ in 2-D, but not in 3-D. Notice that the proof of estimates (2.1) and (2.2) is very similar to that by Ladyzhenskaya [12, pp. 8–11], where it is also proved that the remarkable estimate

$$\|\varphi\|^6_{L^6(\mathbb{R}^3)} \le 48\, \|\nabla\varphi\|^6_{\mathbb{L}^2(\mathbb{R}^3)} \tag{2.6}$$

is valid for any φ with compact support. On the other hand, estimates (2.3) and (2.4) can be viewed as particular cases of Sobolev embedding (or interpolation) inequality, see for example [1].

Lemma 2.2. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be in the spaces* $\mathbb{L}^4(\mathcal{O}, \mathbb{R}^2)$ *and* $\mathbb{V}$, *respectively. Then the following estimate holds*:

$$|\langle B(\mathbf{w}), \mathbf{v}\rangle| \le 2\, \|\mathbf{w}\|^{3/2}\, |\mathbf{w}|^{1/2}\, \|\mathbf{v}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}. \tag{2.7}$$

Proof. In terms of the trilinear form, we have $\langle B(\mathbf{w}), \mathbf{v}\rangle = b(\mathbf{w}, \mathbf{w}, \mathbf{v})$. From estimate (1.13) we deduce

$$|\langle B(\mathbf{w}), \mathbf{v}\rangle| \le \sqrt{2}\, \|\mathbf{w}\|_{\mathbb{V}}\, \|\mathbf{w}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}\, \|\mathbf{v}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}.$$

By means of Lemma 2.1 we get

$$\|\mathbf{w}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)} \le \sqrt[4]{2}\, \|\mathbf{w}\|^{1/2}_{\mathbb{V}}\, \|\mathbf{w}\|^{1/2}_{\mathbb{H}},$$

which completes the proof. □

Notice that in 3-D, we deduce by means of estimate (2.5)

$$|\langle B(\mathbf{w}), \mathbf{v}\rangle| \le 2\, \|\mathbf{w}\|^{7/4}\, |\mathbf{w}|^{1/4}\, \|\mathbf{v}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^3)}, \tag{2.8}$$

instead of (2.7). Actually, it may be better to use the estimate

$$|\langle B(\mathbf{w}), \mathbf{v}\rangle| \le 2\, \|\mathbf{w}\|^{3/2}\, |\mathbf{w}|^{1/2}\, \|\mathbf{v}\|_{\mathbb{L}^6(\mathcal{O},\mathbb{R}^3)}, \tag{2.9}$$

which follows from (2.6) and the interpolation inequality

$$\|\varphi\|_{\mathbb{L}^3(\mathcal{O},\mathbb{R}^3)} \le \|\varphi\|^{1/2}_{\mathbb{L}^2(\mathcal{O},\mathbb{R}^3)}\, \|\varphi\|^{1/2}_{\mathbb{L}^6(\mathcal{O},\mathbb{R}^3)}. \tag{2.10}$$

The above lemma shows that the nonlinear operator $u \mapsto B(u)$ can be considered as mapping the space $\mathbb{V}$ into its dual space $\mathbb{V}'$, so that the compact form of the Navier–Stokes equation (1.3) is meaningful.

Lemma 2.3. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be in the space* $\mathbb{V}$. *Then the following estimates hold*:

$$|\langle B(\mathbf{u}) - B(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle| \le \frac{\nu}{2}\, \|\mathbf{u} - \mathbf{v}\|^2 + \frac{16}{\nu^3}\, |\mathbf{u} - \mathbf{v}|^2\, \|\mathbf{v}\|^4_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}. \tag{2.11}$$

Proof. For given $\mathbf{u}$ and $\mathbf{v}$, set $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Starting with equality (1.12) we get

$$\begin{aligned}\langle B(\mathbf{u}), \mathbf{w}\rangle &= \langle B(\mathbf{u}, \mathbf{u}), \mathbf{w}\rangle = -\langle B(\mathbf{u}, \mathbf{w}), \mathbf{u}\rangle \\ &= -\langle B(\mathbf{u}, \mathbf{w}), \mathbf{w}\rangle - \langle B(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle = -\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle\end{aligned}$$

and

$$\langle B(\mathbf{v}), \mathbf{w}\rangle = -\langle B(\mathbf{v}, \mathbf{w}), \mathbf{v}\rangle,$$

which give

$$\langle B(\mathbf{u}) - B(\mathbf{v}), \mathbf{w}\rangle = -\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle + \langle B(\mathbf{v}, \mathbf{w}), \mathbf{v}\rangle = -\langle B(\mathbf{w}), \mathbf{v}\rangle.$$

Next, by means of Lemma 2.2 we have

$$|\langle B(\mathbf{u}) - B(\mathbf{v}), \mathbf{w}\rangle| \le 2\, \|\mathbf{w}\|^{3/2}\, |\mathbf{w}|^{1/2}\, \|\mathbf{v}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)},$$

and recalling that

$$ab \le \tfrac{3}{4} a^{4/3} + \tfrac{1}{4} b^4, \qquad \forall\, a, b \ge 0,$$

we obtain estimate (2.11). □

At this point, it is clear that the nonlinear operator $u \mapsto Au + B(u)$ is hemi-continuous (actually, continuous) from the Hilbert space $\mathbb{V}$ into its dual $\mathbb{V}'$ since

$$\langle B(\mathbf{u} + \lambda\mathbf{v}), \mathbf{w}\rangle = \langle B(\mathbf{u}), \mathbf{w}\rangle + \lambda\langle B(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, \mathbf{u}), \mathbf{w}\rangle + \lambda^2\langle B(\mathbf{v}), \mathbf{w}\rangle, \tag{2.12}$$

which is continuous in λ. Also, the nonlinear operator $B(\cdot)$ can be considered as a map from $\mathbb{V}$ (respectively, $\mathbb{H}$) into the dual space $\mathbb{V}' \cap \mathbb{L}^{4/3}(\mathcal{O}, \mathbb{R}^2)$ (respectively, $\mathbb{V}' \cap \mathbb{W}^{1,\infty}(\mathcal{O}, \mathbb{R}^2)$). However, $A + B(\cdot)$ is not monotone, but a combination of the previous lemmas lets us deduce the following result.

Lemma 2.4. *For a given $r > 0$ we consider the following (closed) $\mathbb{L}^4$-ball $\mathbb{B}_r$ in the space $\mathbb{V}$:*

$$\mathbb{B}_r := \{\mathbf{v} \in \mathbb{V}\colon\ \|\mathbf{v}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)} \le r\}. \tag{2.13}$$

Then the nonlinear operator $u \mapsto Au + B(u)$ is monotone in the convex ball $\mathbb{B}_r$, i.e., for any $\mathbf{u}$ in $\mathbb{V}$, $\mathbf{v}$ in $\mathbb{B}_r$ and $\mathbf{w} = \mathbf{u} - \mathbf{v}$ we have

$$\langle A\mathbf{w}, \mathbf{w}\rangle + \langle B(\mathbf{u}) - B(\mathbf{v}), \mathbf{w}\rangle + \frac{16 r^4}{\nu^3}\, |\mathbf{w}|^2 \ge \frac{\nu}{2}\, \|\mathbf{w}\|^2. \tag{2.14}$$

Similarly, if $r(t)$ is a positive and measurable real function and $\mathbb{B}_r(t)$ is the following (closed) time-variable $\mathbb{L}^4$-ball of $L^2(0, T; \mathbb{V})$,

$$\mathbb{B}_r(t) := \{\mathbf{v}(\cdot) \in L^2(0, T; \mathbb{V})\colon\ \|\mathbf{v}(t)\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)} \le r(t)\}, \tag{2.15}$$

then for any $\mathbf{u}(\cdot)$ *in* $L^2(0,T;\mathbb{V})$, $\mathbf{v}(t)$ *in* $\mathbb{B}_r(t)$, $\mathbf{w}(\cdot)=\mathbf{u}(\cdot)-\mathbf{v}(\cdot)$ *and any measurable real function* $\rho(t)$, *we have*

$$\int_0^T [\langle A\mathbf{w},\mathbf{w}\rangle + \langle B(\mathbf{u})-B(\mathbf{v}),\mathbf{w}\rangle] e^{\rho(t)}\,dt + \frac{16}{\nu^3}\int_0^T |\mathbf{w}(t)|^2 r^4(t) e^{\rho(t)}\,dt \geq \frac{\nu}{2}\int_0^T \|\mathbf{w}\|^2 e^{\rho(t)}\,dt. \tag{2.16}$$

Proof. This follows from previous results. □

Remark 2.5 (Monotone Quantization). Notice that in [2] a similar type of monotonicity (in a ball of stronger norm) was observed. Actually, if the nonlinearity is modified as follows, $B_r(\cdot)\colon \mathbb{V}\to\mathbb{V}'$,

$$B_r(\mathbf{v}) = \begin{cases} B(\mathbf{v}) & \text{if} \quad \|\mathbf{v}\|_{\mathbb{L}^4} \leq r, \\ \left(\dfrac{r}{\|\mathbf{v}\|_{\mathbb{L}^4}}\right)^4 B(\mathbf{v}) & \text{if} \quad \|\mathbf{v}\|_{\mathbb{L}^4} \geq r, \end{cases} \tag{2.17}$$

then for any $r>0$ there exists a constant $\lambda = 2^{12}r^4/\nu^3$ such that the mapping $\mathbf{v}\mapsto A\mathbf{v}+B_r(\mathbf{v}+\lambda\mathbf{v})$ is monotone. Indeed, consider $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{V}$ and denote by $\mathbb{B}_r$ the (closed) $\mathbb{L}^4$-ball centered at the origin with radius $r>0$. If both $\mathbf{u}$ and $\mathbf{v}$ do not belong to $\mathbb{B}_r$, then for $\mathbf{w}=\mathbf{u}-\mathbf{v}$ we have

$$\langle B_r(\mathbf{u})-B_r(\mathbf{v}),\mathbf{w}\rangle = \left(\frac{r^4}{\|\mathbf{v}\|^4_{\mathbb{L}^4}}\right)\langle B(\mathbf{u})-B(\mathbf{v}),\mathbf{w}\rangle + \left(\frac{r^4}{\|\mathbf{u}\|^4_{\mathbb{L}^4}} - \frac{r^4}{\|\mathbf{v}\|^4_{\mathbb{L}^4}}\right)\langle B(\mathbf{u}),\mathbf{w}\rangle.$$

Since

$$\langle B(\mathbf{u})-B(\mathbf{v}),\mathbf{w}\rangle = -\langle B(\mathbf{w}),\mathbf{v}\rangle,$$

$$\langle B(\mathbf{u}),\mathbf{w}\rangle = -\langle B(\mathbf{u},\mathbf{w}),\mathbf{v}\rangle,$$

$$\frac{r^4}{\|\mathbf{u}\|^4_{\mathbb{L}^4}} - \frac{r^4}{\|\mathbf{v}\|^4_{\mathbb{L}^4}} = r^4\left(\frac{1}{\|\mathbf{u}\|^4_{\mathbb{L}^4}\,\|\mathbf{v}\|_{\mathbb{L}^4}} + \frac{1}{\|\mathbf{u}\|^3_{\mathbb{L}^4}\,\|\mathbf{v}\|^2_{\mathbb{L}^4}} + \frac{1}{\|\mathbf{u}\|^2_{\mathbb{L}^4}\,\|\mathbf{v}\|^3_{\mathbb{L}^4}} + \frac{1}{\|\mathbf{u}\|_{\mathbb{L}^4}\,\|\mathbf{v}\|^4_{\mathbb{L}^4}}\right)(\|\mathbf{v}\|_{\mathbb{L}^4} - \|\mathbf{u}\|_{\mathbb{L}^4})$$

we get

$$|\langle B_r(\mathbf{u})-B_r(\mathbf{v}),\mathbf{w}\rangle| \leq 8r\|\mathbf{w}\|^{3/2}\,||\mathbf{w}|^{1/2}, \tag{2.18}$$

after using estimates (1.13) and (2.2). Similarly, if $\mathbf{u}$ belongs to $\mathbb{B}_r$, but $\mathbf{v}$ does not belong to $\mathbb{B}_r$, then we have

$$\langle B_r(\mathbf{u}) - B_r(\mathbf{v}), \mathbf{w}\rangle = \left(\frac{r^4}{\|\mathbf{v}\|_{\mathbb{L}^4}^4}\right)\langle B(\mathbf{u}) - B(\mathbf{v}), \mathbf{w}\rangle + \left(1 - \frac{r^4}{\|\mathbf{v}\|_{\mathbb{L}^4}^4}\right)\langle B(\mathbf{u}), \mathbf{w}\rangle,$$

and, as above, we deduce estimate (2.18). The case when both $\mathbf{u}$ and $\mathbf{v}$ belong to $\mathbb{B}_r$ is part of the previous lemma. This implies that $A + B_r + \lambda I$ is then *maximal monotone* in $\mathbb{H}$, while the L^p-accretivity of $A + B_r + \lambda I$, for $p \neq 2$, is an open problem.

Now we can prove the following estimate.

Lemma 2.6. *Let $\mathbf{u}(t)$ be a function in $L^2(0, T; \mathbb{V})$ such that $\partial_t \mathbf{u}(t)$ belongs to $L^2(0, T; \mathbb{V}')$ and satisfies the Navier–Stokes equation* (1.3) *with $\mathbf{f}(t)$ in $L^2(0, T; \mathbb{H})$. Then we have the energy equality*

$$|\mathbf{u}(T)|^2 + 2\nu \int_0^T \|\mathbf{u}(t)\|^2 \, dt = |\mathbf{u}(0)|^2 + 2 \int_0^T (\mathbf{f}(t), \mathbf{u}(t)) \, dt, \tag{2.19}$$

which yields the following a priori estimate for any $\varepsilon > 0$:

$$\sup_{0 \le t \le T} |\mathbf{u}(t)|^2 e^{-\varepsilon t} + 2\,\nu \int_0^T \|\mathbf{u}(t)\|^2 e^{-\varepsilon t} \, dt \\ \le |\mathbf{u}(0)|^2 + \frac{1}{\varepsilon}\left(\int_0^T |\mathbf{f}(t)|^2 e^{-\varepsilon t} \, dt\right). \tag{2.20}$$

Moreover, if $\mathbf{f}(t)$ is in $L^4(0, T; \mathbb{H})$, then

$$\sup_{0 \le t \le T} |\mathbf{u}(t)|^4 e^{-\varepsilon t} + 4\,\nu \int_0^T \|\mathbf{u}(t)\|^2 \, |\mathbf{u}(t)|^2 e^{-\varepsilon t} \, dt \\ \le |\mathbf{u}(0)|^4 + \frac{27}{\varepsilon^3}\left(\int_0^T |\mathbf{f}(t)|^4 e^{-\varepsilon t} \, dt\right), \tag{2.21}$$

and if $\mathbf{f}(t)$ belongs $L^4(0, T; \mathbb{H}) \cap \mathbb{L}^4(\mathcal{O} \times (0, T))$, then

$$\sup_{0 \le t \le T} \|\mathbf{u}(t)\|_{\mathbb{L}^4(\mathcal{O})}^4 e^{-\varepsilon t} + 12\,\nu \sum_i \int_0^T |u_i(t) \nabla u_i(t)|^2 e^{-\varepsilon t} \, dt \\ \le \|\mathbf{u}(0)\|_{\mathbb{L}^4(\mathcal{O})}^4 + \frac{27}{\varepsilon^3}\left(\int_0^T \|\mathbf{f}(t) - \nabla p(t)\|_{\mathbb{L}^4(\mathcal{O})}^4 e^{-\varepsilon t} dt\right), \tag{2.22}$$

where $p(t) = p(x, t)$ is the pressure (*scalar*) *field as in* (1.1).

Proof. Indeed, in view of equality (1.12), the elementary inequality

$$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2$$

and the Navier–Stokes equation (1.3), the function

$$F(t) := |\mathbf{u}(t)|^2 e^{-\varepsilon t}$$

satisfies

$$\begin{aligned} F'(t) &= 2(\partial_t \mathbf{u}(t), \mathbf{u}(t)) e^{-\varepsilon t} - \varepsilon |\mathbf{u}(t)|^2 e^{-\varepsilon t} \\ &= -2\nu \|\mathbf{u}(t)\|^2 e^{-\varepsilon t} + 2\langle \mathbf{f}(t), \mathbf{u}(t)\rangle e^{-\varepsilon t} - \varepsilon |\mathbf{u}(t)|^2 e^{-\varepsilon t} \\ &\leq -2\nu \|\mathbf{u}(t)\|^2 e^{-\varepsilon t} + \frac{1}{\varepsilon} |\mathbf{f}(t)|^2 e^{-\varepsilon t}. \end{aligned}$$

Hence, an integration in $[0, T]$ yields (2.20). Similarly, by considering the functions

$$G(t) := |\mathbf{u}(t)|^4 e^{-\varepsilon t} \quad \text{and} \quad H(t) := |\mathbf{u}(t)|^4_{\mathbb{L}^4(\mathcal{O})} e^{-\varepsilon t},$$

and remarking that for $\mathbf{u}^3(x, t) := [u_i^3(x, t)]$ we have

$$4(\partial_t \mathbf{u}(t), \mathbf{u}^3(t)) = \partial_t \|\mathbf{u}(t)\|^4_{\mathbb{L}^4(\mathcal{O})}, \qquad \langle A\mathbf{u}(t), \mathbf{u}^3(t)\rangle = 3 \sum_i |u_i(t) \nabla u_i|^2,$$

and

$$\langle B(\mathbf{u}(t)), \mathbf{u}^3(t)\rangle = 0, \qquad |(\mathbf{f}(t), \mathbf{u}^3(t))| \leq \|\mathbf{f}(t)\|_{\mathbb{L}^4(\mathcal{O})} \|\mathbf{u}(t)\|^{3/4}_{\mathbb{L}^4(\mathcal{O})},$$

we deduce (2.21) and (2.22). □

Notice that in estimate (2.22), the pressure is unknown, but by combining Lemmas 2.1 and 2.6 we obtain an estimate for the norm $L^4(0, T; \mathbb{L}^4(\mathcal{O}; \mathbb{R}^2))$, without using (2.22), which is valid only in dimension 2. In fact it is well known that the pressure term can be estimated by solving an appropriate Poisson problem obtained by applying divergence to the Navier–Stokes equation, e.g., [9]. Essentially, if one can estimate the projection of $\mathbf{u}(t) \cdot \nabla \mathbf{u}(t)$ on $\mathbb{H}^\perp$, then (2.22) yields an estimate on the whole term $\mathbf{u}(t) \cdot \nabla \mathbf{u}(t)$. Moreover, it is possible to use the duality map $|\mathbf{u}|^2 \mathbf{u}$ instead of the expression $\mathbf{u}^3$ to multiply the Navier–Stokes equation.

On the other hand, we can relax the assumption on $\mathbf{f}$ by requesting only that $\mathbf{f}(t)$ belongs to $L^2(0, T; \mathbb{V}')$. In this case, we check that the function $F(t)$ can also be bounded as follows:

$$\begin{aligned} F'(t) &= -2\nu \|\mathbf{u}(t)\|^2 e^{-\varepsilon t} + 2\langle \mathbf{f}(t), \mathbf{u}(t)\rangle e^{-\varepsilon t} - \varepsilon |\mathbf{u}(t)|^2 e^{-\varepsilon t} \\ &\leq -\nu \|\mathbf{u}(t)\|^2 e^{-\varepsilon t} + \frac{1}{\nu} \|\mathbf{f}(t)\|^2_{L^2(0,T;\mathbb{V}')} e^{-\varepsilon t}, \end{aligned}$$

which holds even for $\varepsilon = 0$. This yields the estimate

$$\sup_{0 \leq t \leq T} |\mathbf{u}(t)|^2 + \nu \int_0^T \|\mathbf{u}(t)\|^2 \, dt \leq |\mathbf{u}(0)|^2 + \frac{1}{\nu} \left(\int_0^T \|\mathbf{f}(t)\|^2_{L^2(0,T;\mathbb{V}')} \, dt \right). \tag{2.23}$$

Similarly, if $\mathbf{f}(t)$ is in $L^4(0, T; \mathbb{V}')$, then

$$\sup_{0\le t\le T} |\mathbf{u}(t)|^4 e^{-\varepsilon t} + 2\,\nu \int_0^T \|\mathbf{u}(t)\|^2\, |\mathbf{u}(t)|^2 e^{-\varepsilon t}\, dt$$
$$\le |\mathbf{u}(0)|^4 + \frac{2}{\nu^2 \varepsilon} \left(\int_0^T \|\mathbf{f}(t)\|^4_{L^2(0,T;\mathbb{V}')} e^{-\varepsilon t}\, dt \right), \tag{2.24}$$

for any $\varepsilon > 0$.

By means of estimate (2.8) we check that the above estimate (2.11) and Lemmas 2.4 and 2.6 remain true in 3-D. However, estimate (2.21) is not sufficient to ensure a bound in the space $L^4(\Omega \times (0, T))$, since we need to bound the $\mathbb{V}$-norm in $L^3(0, T)$, see estimate (2.5).

Remark 2.7. In general, if $\mathbf{u}(t)$ belongs to $\mathbb{H}\cap\mathbb{H}^2(\mathcal{O}, \mathbb{R}^2)$, then $\triangle\mathbf{u}(t)$ ($\nabla\mathbf{u}(t)$, respectively) does not necessarily belong to $\mathbb{H}$ ($\mathbb{V}$, respectively). However, the norms $|\triangle \cdot|$ ($|\nabla \cdot|$, respectively) and $|A \cdot|$ ($|A^{1/2} \cdot|$, respectively) are equivalent (for instance, we refer to [21] for details and more comments). Let $\mathbf{u}(t)$ satisfy the Navier–Stokes equation (1.3) with $\mathbf{f}(t)$ in $L^2(0, T; \mathbb{H})$. If $\mathbf{u}(t)$ belongs to $L^2(0, T; \mathbb{H}^2(\mathcal{O}, \mathbb{R}^2))$ and $\partial_t\mathbf{u}(t)$ belongs to $L^2(0, T; \mathbb{H})$, then multiplying (1.3) by $-P_{\mathbb{H}}\triangle\mathbf{u}(t)$ we have

$$\tfrac{1}{2}\partial_t |\nabla\mathbf{u}(t)|^2 + \nu |P_{\mathbb{H}}\triangle\mathbf{u}(t)|^2 = \langle B(\mathbf{u}(t)), \triangle\mathbf{u}(t)\rangle - (\mathbf{f}(t), \triangle\mathbf{u}(t)), \tag{2.25}$$

after recalling that here $P_{\mathbb{H}}\mathbf{f}(t) = \mathbf{f}(t)$. Since $|P_{\mathbb{H}}\triangle\mathbf{u}|$ is equivalent to $|\triangle\mathbf{u}|$, there is a constant $c > 0$ such that

$$|\langle B(\mathbf{u}), \triangle\mathbf{u}\rangle| \le 2|\triangle\mathbf{u}|\, |\nabla\mathbf{u}|_{\mathbb{L}^4}\, |\mathbf{u}|_{\mathbb{L}^4} \le c|\triangle\mathbf{u}|^{3/2}\, |\nabla\mathbf{u}|^{1/2}\, |\mathbf{u}|_{\mathbb{L}^4} \tag{2.26}$$

and we obtain

$$\partial_t |\nabla\mathbf{u}(t)|^2 + \nu |P_{\mathbb{H}}\triangle\mathbf{u}(t)|^2 \le c_\nu\, |\mathbf{u}(t)|^4_{\mathbb{L}^4} |\nabla\mathbf{u}(t)|^2 + |\mathbf{f}(t)|\, |\triangle\mathbf{u}(t)|, \tag{2.27}$$

for some constant c_ν. Then an estimate on $\mathbf{u}(t)$ in the spaces $L^\infty(0, T; \mathbb{V})$ and $L^2(0, T; \mathbb{H}^2(\mathcal{O}, \mathbb{R}^2))$ is established.

3. Stochastic PDE

Here we look at the compact formulation (1.3) of the Navier–Stokes equation subject to a random (Gaussian) term, i.e., the forcing field $\mathbf{f}$ has a mean value still denoted by $\mathbf{f}$ and a noise denoted by $\dot{\mathbf{G}}$. We can write $\mathbf{f}(t) = \mathbf{f}(t, x)$ and the noise process $\dot{\mathbf{G}}(t) = \dot{\mathbf{G}}(t, x)$ as a series $d\mathbf{G}_k = \sum_k \mathbf{g}_k\, dw_k$, where $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, \ldots)$ and $w = (w_1, w_2, \ldots)$ are regarded as ℓ^2-valued functions. The stochastic noise process represented by $\mathbf{g}\, dw(t) = \sum_k \mathbf{g}_k(t, x)\, dw_k(t, \omega)$ (notice that most of the time we omit the variable ω) is normal distributed in $\mathbb{H}$ with a trace-class co-variance operator denoted by $\mathbf{g}^*\mathbf{g}(t)$ and given by

$$\begin{cases} (\mathbf{g}^*\mathbf{g}(t)\mathbf{u}, \mathbf{v}) := \sum_k (\mathbf{g}_k(t), \mathbf{u})\, (\mathbf{g}_k(t), \mathbf{v}), \\ \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t)) := \sum_k |\mathbf{g}_k(t)|^2 < \infty, \end{cases} \tag{3.1}$$

i.e., the mapping (stochastic integral) induced by the noise

$$\mathbf{v} \mapsto \int_0^T (\mathbf{g}(t)\, dw(t), \mathbf{v}) := \sum_k \int_0^T (\mathbf{g}_k(t), \mathbf{v})\, dw_k(t) \tag{3.2}$$

is a continuous linear functional on $\mathbb{H}$ with probability 1 and the noise is the formal time-derivative of the Gaussian process $\mathbf{G}(t) = \int_0^t \mathbf{g}(t)\, dw(t)$.

We interpret the stochastic Navier–Stokes equation as an Itô stochastic equation in variational form

$$d(\mathbf{u}(t), \mathbf{v}) + \langle A\mathbf{u}(t) + B(\mathbf{u}(t)), \mathbf{v}\rangle\, dt = (\mathbf{f}(t), \mathbf{v})\, dt + \sum_k (\mathbf{g}_k(t), \mathbf{v})\, dw_k(t), \tag{3.3}$$

in $(0, T)$, with the initial condition

$$(\mathbf{u}(0), \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}), \tag{3.4}$$

for any $\mathbf{v}$ in the space $\mathbb{V}$. This requires the following assumption on the data:

$$\mathbf{f} \in L^2(0, T; \mathbb{V}), \qquad \mathbf{g} \in L^2(0, T; \ell_2(\mathbb{H})), \qquad \mathbf{u}_0 \in \mathbb{H}, \tag{3.5}$$

and we expect a solution as an adapted (and measurable) stochastic process $\mathbf{u} = \mathbf{u}(t, x, \omega)$ satisfying

$$\mathbf{u} \in L^2(\Omega; C^0(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V})) \tag{3.6}$$

and the (linear) energy equality

$$\begin{aligned} &d|\mathbf{u}(t)|^2 + 2\nu\, |\nabla \mathbf{u}(t)|^2\, dt \\ &\quad = \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))\, dt + 2\, (\mathbf{f}(t), \mathbf{u}(t))\, dt + 2 \sum_k (\mathbf{g}_k(t), \mathbf{u}(t))\, dw_k(t), \end{aligned} \tag{3.7}$$

where we have used the estimate

$$E\left\{ \left| \int_0^T (\mathbf{g}(t)\, dw(t), \mathbf{v}(t)) \right|^2 \right\} \leq \left(\int_0^T \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))\, dt \right) \left(\sup_{0 \leq t \leq T} E|\mathbf{v}(t)|^2 \right) \tag{3.8}$$

for any adapted process $\mathbf{v}$ with values in $L^\infty(0, T; \mathbb{H})$, to make the stochastic integral meaningful. Actually, a more general martingale estimate holds, namely

$$\begin{aligned} &E\left\{ \sup_{0 \leq t \leq T} \left| \int_0^t (\mathbf{g}\, dw(s), \mathbf{v}(s)) \right|^p \right\} \\ &\quad \leq C_p\, E\left\{ \left(\int_0^T \sum_k (\mathbf{g}_k(t), \mathbf{v}(t))^2\, dt \right)^{p/2} \right\}, \end{aligned} \tag{3.9}$$

for any $1 \leq p < \infty$ and some constant C_p depending only on p, e.g., we may take $C_2 = 2$ and $C_1 = 3$.

Moreover, if we also assume that

$$\mathbf{f} \in L^4(0, T; \mathbb{L}^4(\mathcal{O})), \qquad \mathbf{g} \in L^4(0, T; \ell_2(\mathbb{L}^4(\mathcal{O}))), \qquad \mathbf{u}_0 \in \mathbb{L}^4(\mathcal{O}), \tag{3.10}$$

then we have the (linear) $\mathbb{L}^4$-energy equality

$$d\|\mathbf{u}(t)\|^4_{\mathbb{L}^4(\mathcal{O})} + 12\,\nu \sum_i |u_i(t)\nabla u_i(t)|^2\,dt$$
$$= \sum_k (\mathbf{g}_k(t), \mathbf{u}(t))^2\,dt + 4\,(\mathbf{f}(t) - \nabla p(t), \mathbf{u}^3(t))\,dt$$
$$+ 4 \sum_k (\mathbf{g}_k(t), \mathbf{u}^3(t))\,dw_k(t), \tag{3.11}$$

where $\mathbf{u}^3(x,t,\omega) := [u_i^3(x,t,\omega)]$ and $p = p(x,t,\omega)$ is the pressure. As mentioned before, the pressure (scalar) field p is unknown, so that equality (3.11) is of limited help.

A finite-dimensional (Galerkin) approximation of the stochastic Navier–Stokes equation (3.3) can be defined as follows. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$ be a complete orthonormal system (i.e., a basis) in the Hilbert space $\mathbb{H}$ belonging to the space $\mathbb{V}$ (and $\mathbb{L}^4$). Denote by $\mathbb{H}_n$ the n-dimensional subspace of $\mathbb{H}$ and $\mathbb{V}$ of all linear combinations of the first n elements $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Consider the following stochastic ODE in $\mathbb{H}_n$ (i.e., essentially in $\mathbb{R}^n$):

$$d(\mathbf{u}^n(t), \mathbf{v}) + \langle A\mathbf{u}^n(t) + B(\mathbf{u}^n(t)), \mathbf{v}\rangle\,dt$$
$$= (\mathbf{f}(t), \mathbf{v})\,dt + \sum_k (\mathbf{g}_k(t), \mathbf{v})\,dw_k(t), \tag{3.12}$$

in $(0, T)$, with the initial condition

$$(\mathbf{u}(0), \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}), \tag{3.13}$$

for any $\mathbf{v}$ in the space $\mathbb{H}_n$. The coefficients involved are locally Lipschitz, so that we need some a priori estimate to show the global existence of a solution $\mathbf{u}^n(t)$ as an adapted process in the space $C^0(0, T, \mathbb{H}_n)$.

Proposition 3.1 (Energy Estimate). *Assume the data* $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{u}_0$ *satisfying condition* (3.5). *Let* $\mathbf{u}^n(t)$ *be an adapted process in the space* $C^0(0, T, \mathbb{H}_n)$ *solution of the stochastic ODE* (3.12). *Then we have the energy equality*

$$d|\mathbf{u}^n(t)|^2 + 2\nu\,|\nabla\mathbf{u}^n(t)|^2 dt$$
$$= [2\,(\mathbf{f}(t), \mathbf{u}^n(t)) + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))]\,dt + 2\sum_k (\mathbf{g}_k(t), \mathbf{u}^n(t))\,dw_k(t), \tag{3.14}$$

which yields the following a priori estimate for any $\varepsilon > 0$:

$$E\{|\mathbf{u}^n(t)|^2\}e^{-\varepsilon t} + 2\,\nu \int_0^T E\{|\nabla\mathbf{u}^n(t)|^2\}e^{-\varepsilon t}\,dt$$
$$\leq |\mathbf{u}(0)|^2 + \int_0^T \left[\frac{1}{\varepsilon}|\mathbf{f}(t)|^2 + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))\right] e^{-\varepsilon t}\,dt, \tag{3.15}$$

for any $0 \le t \le T$. *Moreover, if we suppose*

$$\mathbf{f} \in L^p(0, T; \mathbb{H}), \qquad \mathbf{g} \in L^p(0, T; \ell_2(\mathbb{H})), \tag{3.16}$$

then we also have

$$E\left\{\sup_{0\le t\le T} |\mathbf{u}^n(t)|^p e^{-\varepsilon t} + p\,\nu \int_0^T |\nabla \mathbf{u}^n(t)|^2 |\mathbf{u}^n(t)|^{p-2} e^{-\varepsilon t}\,dt\right\}$$
$$\le |\mathbf{u}(0)|^p + C_{\varepsilon,p,T}\int_0^T [|\mathbf{f}(t)|^p + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))^{p/2}]e^{-\varepsilon t}\,dt, \tag{3.17}$$

for some constant $C_{\varepsilon,p,T}$ *depending only on* $\varepsilon > 0$, $1 \le p < \infty$ *and* $T > 0$.

Proof. Indeed, we notice first that (3.12) implies that

$$d\mathbf{u}^n(t) + [A^n\mathbf{u}^n(t) + B^n(\mathbf{u}^n(t))]\,dt = \mathbf{f}^n(t)\,dt + \sum_k \mathbf{g}_k^n(t)\,dw_k(t), \tag{3.18}$$

where A^n, $B^n(\cdot)$, $\mathbf{f}^n(t)$ and $\mathbf{g}_k^n(t)$ are the orthogonal projection on the finite-dimensional subspace $\mathbb{H}_n'$, the dual space of $\mathbb{H}_n$. Hence, by using Itô's formula with the process $\mathbf{u}^n(t)$ and the function $u \mapsto |u|^2$, we obtain the energy equality (3.14) after noticing that $(B^n(\mathbf{u}^n(t)), \mathbf{u}^n(t)) = (B(\mathbf{u}^n(t)), \mathbf{u}^n(t)) = 0$.

Next, as in Lemma 2.6 we calculate the stochastic differential of the process $F(t) := |\mathbf{u}^n(t)|^2 e^{-\varepsilon t}$ to get

$$\begin{aligned} dF(t) &= e^{-\varepsilon t}\,d|\mathbf{u}^n(t)|^2 - \varepsilon|\mathbf{u}^n(t)|^2 e^{-\varepsilon t}\,dt \\ &= -2\nu\|\mathbf{u}^n(t)\|^2 e^{-\varepsilon t}\,dt + [2\,(\mathbf{f}(t), \mathbf{u}^n(t)) + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))]e^{-\varepsilon t}\,dt \\ &\quad + 2\sum_k (\mathbf{g}_k(t), \mathbf{u}^n(t))e^{-\varepsilon t}\,dw_k(t), \end{aligned}$$

which yields the a priori estimate (3.15).

Similarly, consider $G(t) := |\mathbf{u}^n(t)|^p e^{-\varepsilon t}$ and use Itô calculus based on the energy process $|\mathbf{u}^n(t)|^2$. As in Lemma 2.6, we check that its stochastic differential satisfies

$$\begin{aligned} &dG(t) + p\,\nu\|\mathbf{u}^n(t)\|^2|\mathbf{u}^n(t)|^{p-2}e^{-\varepsilon t}\,dt + \varepsilon|\mathbf{u}^n(t)|^p e^{-\varepsilon t}\,dt \\ &\quad = \Big[p\,(\mathbf{f}(t), \mathbf{u}^n(t)) + \frac{p}{2}\,\mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))\Big]|\mathbf{u}^n(t)|^{p-2}e^{-\varepsilon t}\,dt \\ &\qquad + \frac{p(p-2)}{8}\sum_k (\mathbf{g}_k(t), \mathbf{u}^n(t))^2|\mathbf{u}^n(t)|^{p-4}e^{-\varepsilon t}\,dt \\ &\qquad + p\sum_k (\mathbf{g}_k(t), \mathbf{u}^n(t))|\mathbf{u}^n(t)|^{p-2}e^{-\varepsilon t}\,dw_k(t). \end{aligned}$$

Hence, by means of the elementary inequality

$$ab \le \frac{a^p}{p} + \frac{a^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad ab > 0,$$

there is a constant $C_{\varepsilon,p}$ depending only on $\varepsilon > 0$ and $1 \le p < \infty$ such that

$$\begin{aligned} dG(t) &+ p\,\nu \|\mathbf{u}^n(t)\|^2 |\mathbf{u}^n(t)|^{p-2} e^{-\varepsilon t}\, dt + \frac{\varepsilon}{2} |\mathbf{u}^n(t)|^p e^{-\varepsilon t}\, dt \\ &\le C_{\varepsilon,p} [|\mathbf{f}(t)|^p + \mathrm{Tr}(\mathbf{g}^* \mathbf{g}(t))^{p/2}] e^{-\varepsilon t}\, dt \\ &\quad + p \sum_k (\mathbf{g}_k(t), \mathbf{u}^n(t)) |\mathbf{u}^n(t)|^{p-2} e^{-\varepsilon t}\, dw_k(t). \end{aligned} \tag{3.19}$$

This yields the p-bound

$$\begin{aligned} E\{|\mathbf{u}^n(t)|^p\} e^{-\varepsilon t} &+ p\,\nu \int_0^T E\{\|\mathbf{u}^n(t)\|^2 |\mathbf{u}^n(t)|^{p-2}\} e^{-\varepsilon t}\, dt \\ &\le |\mathbf{u}(0)|^p + C_{\varepsilon,p} \int_0^T [|\mathbf{f}(t)|^p + \mathrm{Tr}(\mathbf{g}^* \mathbf{g}(t))^{p/2}] e^{-\varepsilon t}\, dt, \end{aligned} \tag{3.20}$$

for any $0 \le t \le T$.

On the other hand, if after integrating the stochastic differential (3.19) but before taking the mathematical expectation we calculate the sup norm in $[0, T]$, then we have to deal with a term of the form

$$A := E\left\{ \sup_{0 \le t \le T} \left| \int_0^t \sum_k (\mathbf{g}_k(s), \mathbf{u}^n(s)) |\mathbf{u}^n(s)|^{p-2} e^{-\varepsilon s}\, dw_k(s) \right| \right\}.$$

By means of the martingale inequality (3.9) for $p = 1$, we deduce

$$\begin{aligned} A &\le C_1 E\left\{ \left(\int_0^T \sum_k (\mathbf{g}_\mathbf{k}(t), \mathbf{u}^n(t))^2 |\mathbf{u}^n(t)|^{2p-4} e^{-2\varepsilon t}\, dt \right)^{1/2} \right\} \\ &\le C_1 E\left\{ \left(\int_0^T \mathrm{Tr}(\mathbf{g}^* \mathbf{g}(t)) |\mathbf{u}^n(t)|^{2p-2} e^{-2\varepsilon t}\, dt \right)^{1/2} \right\} \\ &\le C_1 E\left\{ \sup_{0 \le t \le T} (|\mathbf{u}^n(t)|^{p-1} e^{-\varepsilon t/p'}) \left(\int_0^T \mathrm{Tr}(\mathbf{g}^* \mathbf{g}(t)) e^{-(2/p)\varepsilon t}\, dt \right)^{1/2} \right\} \\ &\le \frac{\varepsilon}{2} E\left\{ \sup_{0 \le t \le T} (|\mathbf{u}^n(t)|^p e^{-\varepsilon t}) \right\} + C_{\varepsilon,p,T} E\left\{ \int_0^T \mathrm{Tr}(\mathbf{g}^* \mathbf{g}(t))^{p/2} e^{-\varepsilon t}\, dt \right\}, \end{aligned}$$

where the constant $C_{\varepsilon,p,T}$ depends only on $\varepsilon > 0$, $1 \le p < \infty$ and $T > 0$. This provides the estimate (3.17). □

Similarly, we may relax the assumption on the data by requesting only that **f** belongs to $L^2(0, T; \mathbb{V}')$. In this case we have the estimate

$$\begin{aligned} E\{|\mathbf{u}^n(t)|^2\} &+ \nu \int_0^T E\{\|\mathbf{u}^n(t)\|^2\}\, dt \\ &\le |\mathbf{u}(0)|^2 + \int_0^T \left[\frac{1}{\nu} \|\mathbf{f}(t)\|_{\mathbb{V}'}^2 + \mathrm{Tr}(\mathbf{g}^* \mathbf{g}(t)) \right] dt \end{aligned} \tag{3.21}$$

for any $0 \le t \le T$. Moreover, if we suppose

$$\mathbf{f} \in L^p(0,T;\mathbb{V}'), \qquad \mathbf{g} \in L^p(0,T;\ell_2(\mathbb{H})), \tag{3.22}$$

then we also have

$$E\left\{\sup_{0\le t\le T} |\mathbf{u}^n(t)|^p e^{-\varepsilon t} + \frac{p}{2}\nu \int_0^T |\nabla \mathbf{u}^n(t)|^2 |\mathbf{u}^n(t)|^{p-2} e^{-\varepsilon t}\,dt\right\}$$
$$\le |\mathbf{u}(0)|^p + C_{\varepsilon,p,T,\nu}\int_0^T [\|\mathbf{f}(t)\|_{\mathbb{V}'}^p + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))^{p/2}] e^{-\varepsilon t}\,dt, \tag{3.23}$$

for some constant $C_{\varepsilon,p,T,\nu}$ depending only on $\varepsilon > 0$, $1 \le p < \infty$, $T > 0$ and $\nu > 0$. Actually, because the domain $\mathcal{O}$ is bounded, the above estimate remains true for $\varepsilon = 0$.

Under condition (3.5), the stochastic Navier–Stokes equation (3.3) or its finite-dimensional approximation (3.12) is meaningful for adapted processes $\mathbf{u}(t,x,\omega)$ satisfying

$$\mathbf{u} \in L^2(0,T;\mathbb{V}), \qquad B(\mathbf{u}) \in L^2(0,T;\mathbb{V}'), \tag{3.24}$$

with probability 1. As mentioned before, for 2-D we actually have $B(\mathbf{u})$ in $L^2(0,T;\mathbb{V}')$ if $\mathbf{u}$ belongs to $L^2(0,T;\mathbb{V})$. In general, if we only have $\mathbf{u}(\cdot)$ in $L^2(0,T;\mathbb{V})$, then we are allowed to evaluate (3.3) for $\mathbf{v}$ in a dense subspace of $\mathbb{V}$ where $\langle B(\mathbf{w}),\mathbf{v}\rangle$ is defined for any $\mathbf{w}$ in $\mathbb{V}$, e.g., for $\mathbf{v}$ a test function in $\mathcal{O}$.

Now we deal with the uniqueness of the SPDE (3.3) and, in particular, with the finite-dimensional approximation (3.12).

Proposition 3.2 (Uniqueness). *Let* $\mathbf{u}$ *be a solution of the SPDE* (3.3), *i.e., an adapted stochastic process* $\mathbf{u}(t,x,\omega)$ *satisfying* (3.3), (3.4) *with the regularity*

$$\mathbf{u} \in L^2(\Omega; C^0(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})), \qquad \mathbf{u} \in \mathbb{L}^4(\mathcal{O}\times(0,T)\times\Omega), \tag{3.25}$$

and where the data $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{u}_0$ *satisfy condition*

$$\mathbf{f} \in L^2(0,T;\mathbb{V}'), \qquad \mathbf{g} \in L^2(0,T;\ell_2(\mathbb{H})), \qquad \mathbf{u}_0 \in \mathbb{H}. \tag{3.26}$$

If $\mathbf{v}$ *is another solution of the stochastic Navier–Stokes equation* (3.3) *as an adapted stochastic process in the space* $C^0(0,T,\mathbb{H}) \cap L^2(0,T,\mathbb{V})$, *then*

$$|\mathbf{u}(t)-\mathbf{v}(t)|^2 \exp\left[-\frac{32}{\nu^3}\int_0^t \|\mathbf{u}(s)\|_{\mathbb{L}^4(\mathcal{O})}^4\,ds\right] \le |\mathbf{u}(0)-\mathbf{v}(0)|^2, \tag{3.27}$$

with probability 1, *for any* $0 \le t \le T$. *In particular* $\mathbf{u} = \mathbf{v}$, *if* $\mathbf{v}$ *satisfies the initial condition* (3.4).

Proof. Indeed, we notice that if $\mathbf{u}$ and $\mathbf{v}$ are two solutions, then $\mathbf{w} = \mathbf{v} - \mathbf{u}$ solves the deterministic equation

$$\partial_t \mathbf{w} + A\mathbf{w} = B(\mathbf{u}) - B(\mathbf{v}) \qquad \text{in } \mathbb{L}^2(0,T;\mathbb{V}').$$

Setting

$$r(t) := \frac{32}{\nu^3} \int_0^t \|\mathbf{u}(\mathbf{s})\|^4_{\mathbb{L}^4(\mathcal{O})}\, ds$$

and using Lemma 2.4 we have

$$\begin{aligned} & d(e^{-r(t)}|\mathbf{w}(t)|^2) + \nu e^{-r(t)}\|\mathbf{w}(t)\|^2 dt \\ & \quad = -\dot{r}(t)e^{-r(t)}|\mathbf{w}(t)|^2\, dt - \nu e^{-r(t)}\|\mathbf{w}(t)\|^2\, dt \\ & \qquad - 2e^{-r(t)}\langle B(\mathbf{v}(t)) - B(\mathbf{u}(t)), \mathbf{w}(t)\rangle\, dt \leq 0. \end{aligned}$$

Hence, integrating in t, we deduce (3.27), with probability 1. □

Notice that a solution $\mathbf{u}$ of the stochastic Navier–Stokes equation (3.3) in the space $L^2(\Omega; L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V}))$ actually belongs to a better space, namely $L^2(\Omega; C^0(0,T;\mathbb{H}) \cap \mathbb{L}^4(\mathcal{O} \times (0,T)))$, with $\mathcal{O} \subset \mathbb{R}^2$. Thus in 2-D, the uniqueness holds in the space $L^2(\Omega; L^2(0,T;\mathbb{V}))$. Clearly, this also applies to the finite-dimensional approximation (3.12) in the space $L^2(\Omega; L^2(0,T;\mathbb{H}_n))$, but it is not needed there since the coefficients are locally Lipschitz in $\mathbb{H}_n$. We also note that an argument similar to the above was used in [19] for the uniqueness of solutions with multiplicative noise.

Let $r(t,\omega)$ be the integral on $[0,t]$ of an adapted, non-negative and integrable stochastic process $\dot{r}(t,\omega)$. It is clear that for any (adapted process) solution $\mathbf{u}$ of the stochastic Navier–Stokes equation (3.3) such that

$$\mathbf{u} \in L^2(\Omega; L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})), \tag{3.28}$$

the new process $\bar{\mathbf{u}} := \mathbf{u}e^{-r}$ satisfies

$$\begin{aligned} & d(\bar{\mathbf{u}}(t), \mathbf{v}) + \langle A\bar{\mathbf{u}}(t) + e^{-r(t)}B(\bar{\mathbf{u}}(t)) + \dot{r}(t)\bar{\mathbf{u}}(t), \mathbf{v}\rangle\, dt \\ & \quad = (e^{-r(t)}\mathbf{f}(t), \mathbf{v})\, dt + \sum_k (e^{-r(t)}\mathbf{g}_k(t), \mathbf{v})\, dw_k(t), \end{aligned} \tag{3.29}$$

for any function $\mathbf{v}$ in $\mathbb{V} \cap \mathbb{L}^\infty(\mathcal{O})$. Conversely, if $\bar{\mathbf{u}}(t)$ is any (adapted process) solution of (3.29), such that $\mathbf{u} := \bar{\mathbf{u}}e^r$ satisfies (3.28), then $\mathbf{u}$ is indeed a solution of the stochastic Navier–Stokes equation (3.3).

Regarding the energy equality, we remark that for a given adapted process $\mathbf{u}(x,t,\omega)$ in $L^2(\Omega; L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V}))$ satisfying

$$d(\mathbf{u}(t), \mathbf{v}) = \langle \mathbf{h}(t), \mathbf{v}\rangle\, dt + (\mathbf{g}(t), \mathbf{v})\, dw(t), \tag{3.30}$$

for any function $\mathbf{v}$ in $\mathbb{V}$ and some $\mathbf{h}$ in $L^2(0,T;\mathbb{V}')$ and $\mathbf{g}$ in $L^2(0,T;\ell_2(\mathbb{H}))$, we can find a version of $\mathbf{u}$ (still denoted by $\mathbf{u}$) in the space $L^2(\Omega; C^0(0,T;\mathbb{H}))$, and the energy equality

$$d|\mathbf{u}(t)|^2 = [2\langle \mathbf{h}(t), \mathbf{u}(t)\rangle + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t)]\, dt + 2(\mathbf{g}(t), \mathbf{u}(t))\, dw(t) \tag{3.31}$$

holds, for instance see [11]. In our context, any solution of the stochastic Navier–Stokes equation (3.3) satisfying (3.28) has a continuous version, i.e., in the space

$L^2(\Omega, C^0([0,T],\mathbb{H}))$ such that the energy equality

$$d|\mathbf{u}(t)|^2 + 2\nu\,|\nabla\mathbf{u}(t)|^2\,dt = \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t)\,dt + 2\,(\mathbf{f}(t),\mathbf{u}(t))\,dt + 2\sum_k(\mathbf{g}_k(t),\mathbf{u}(t))\,dw_k(t), \tag{3.32}$$

i.e., (3.7), holds.

Proposition 3.3 (2-D Existence). *Let* $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{u}_0$ *be such that*

$$\mathbf{f}\in L^p(0,T;\mathbb{V}'),\qquad \mathbf{g}\in L^p(0,T;\ell_2(\mathbb{H})),\qquad \mathbf{u}_0\in\mathbb{H}, \tag{3.33}$$

for some $p\geq 4$. *Then there exists an adapted process* $\mathbf{u}(t,x,\omega)$ *with the regularity*

$$\mathbf{u}\in L^p(\Omega;C^0(0,T;\mathbb{H}))\cap L^2(\Omega;L^2(0,T;\mathbb{V}))\cap\mathbb{L}^4(\mathcal{O}\times(0,T)\times\Omega) \tag{3.34}$$

and satisfying (3.3), (3.4).

Proof. Indeed, denoting by $F(\mathbf{u})$ the operator $\nu A\mathbf{u}+B(\mathbf{u})-\mathbf{f}$ we have

$$d\mathbf{u}^n(t) + F(\mathbf{u}^n(t))\,dt = \mathbf{g}\,dw(t),\qquad \text{in } \mathbb{H}_n',$$

and based on the a priori estimates (3.23) we can extract a subsequence such that

$$\mathbf{u}^n \longrightarrow \mathbf{u}\qquad \text{weakly star in } L^p(\Omega;L^\infty(0,T;\mathbb{H}))\cap L^2(\Omega;L^2(0,T;\mathbb{V})),$$
$$F(\mathbf{u}^n)\longrightarrow F_0\qquad \text{weakly in } L^2(\Omega;L^2(0,T;\mathbb{V}')),$$

where $\mathbf{u}$ has the Itô differential

$$d\mathbf{u}(t) + F_0(t)\,dt = \mathbf{g}(t)\,dw(t),$$

in $L^2(\Omega;L^2(0,T;\mathbb{V}'))$, and the energy equality holds, i.e.,

$$d|\mathbf{u}(t)|^2 + 2\langle F_0(t),\mathbf{u}(t)\rangle\,dt = \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))\,dt + 2(\mathbf{g}(t),\mathbf{u}(t))\,dw(t).$$

Notice that we also have $|\mathbf{u}^n(0)-\mathbf{u}(0)|$ goes to 0 in $L^2(\Omega)$ and that $\mathbf{u}^n$ converges to $\mathbf{u}$ weakly star in the Banach space $L^p(\Omega;C^0(0,T;\mathbb{H}))$ so that $t\mapsto\mathbf{u}(t)$ is a continuous function from $[0,T]$ into $\mathbb{H}$ with probability 1.

Now, for any adapted process $\mathbf{v}(t,x,\omega)$ in $L^\infty((0,T)\times\Omega;\mathbb{H}_m)$, with $m\leq n$ we define

$$r(t,\omega) := \frac{32}{\nu^3}\int_0^t \|\mathbf{v}(\mathbf{s},\cdot,\omega)\|^4_{\mathbb{L}^4(\mathcal{O})}\,ds$$

as an adapted, continuous (and bounded in ω) real-valued process in $[0,T]$. From the energy equality

$$E\{de^{-r(t)}|\mathbf{u}^n(t)|^2 + e^{-r(t)}\langle 2F(\mathbf{u}^n(t)) + \dot r(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle\,dt\} = E\{e^{-r(t)}\,\mathrm{Tr}(\mathbf{g_n}^*\mathbf{g_n}(t)\,dt\},$$

the fact that the initial condition $\mathbf{u}^n(0)$ converges in $\mathbb{L}^2$, and the lower-semi-continuity of the $\mathbb{L}^2$-norm, we deduce

$$\begin{aligned}
&\liminf_n E\left\{-\int_0^T e^{-r(t)}\langle 2F(\mathbf{u}^n(t))+\dot r(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle\,dt\right\}\\
&\quad=\liminf_n E\left\{e^{-r(T)}|\mathbf{u}^n(T)|^2-|\mathbf{u}^n(0)|^2-\int_0^T e^{-r(t)}\ \mathrm{Tr}(\mathbf{g_n}^*\mathbf{g_n}(t))\,dt\right\}\\
&\quad\geq E\left\{e^{-r(T)}|\mathbf{u}(T)|^2-|\mathbf{u}(0)|^2-\int_0^T e^{-r(t)}\ \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))\,dt\right\}\\
&\quad=E\left\{-\int_0^T e^{-r(t)}\langle 2F_0(t)+\dot r(t)\mathbf{u}(t),\mathbf{u}(t)\rangle\,dt\right\}.
\end{aligned}$$

Next, in view of Lemma 2.4 (monotonicity on $\mathbb{L}^4$-balls) we have

$$\begin{aligned}
&E\left\{\int_0^T e^{-r(t)}\langle 2F(\mathbf{v}(t))+\dot r(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}^n(t)\rangle\,dt\right\}\\
&\quad\geq E\left\{\int_0^T e^{-r(t)}\langle 2F(\mathbf{u}^n(t))+\dot r(t)\mathbf{u}^n(t),\mathbf{v}(t)-\mathbf{u}^n(t)\rangle\,dt\right\},
\end{aligned}$$

and taking limit in n we obtain

$$\begin{aligned}
&E\left\{\int_0^T e^{-r(t)}\langle 2F(\mathbf{v}(t))+\dot r(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle\,dt\right\}\\
&\quad\geq E\left\{\int_0^T e^{-r(t)}\langle 2F_0(t)+\dot r(t)\mathbf{u}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle\,dt\right\}.
\end{aligned}\tag{3.35}$$

Since this last inequality holds for every $\mathbf{v}$ in $L^\infty((0,T)\times\Omega;\mathbb{H}_m)$ and any m, a density argument show that (3.35) remains true for any adapted process $\mathbf{v}$ in $L^6(\Omega;L^\infty(0,T;\mathbb{H}))\cap L^2(\Omega;L^2(0,T;\mathbb{V}))$ such that

$$E\left\{\int_0^T \|\mathbf{v}(t,\cdot,\omega)\|^4_{\mathbb{L}^4(\mathcal{O})}\,|\mathbf{v}(t,\cdot,\omega)-\mathbf{u}(t,\cdot,\omega)|^2\,dt\right\}<\infty.$$

In 2-D we can control the $\mathbb{L}^4(\mathcal{O}\times(0,T))$-norm with the norms in the spaces $L^\infty(0,T;\mathbb{H})$ and $L^2(0,T;\mathbb{V})$, see (2.2) of Lemma 2.1, so that the process $\mathbf{u}$ satisfies the above condition. In 3-D we may compare (2.2) with estimate (2.5), where only the $L^3(0,T;\mathbb{L}^4(\mathcal{O}))$ can be bounded.

Hence, first we take $\mathbf{v}:=\mathbf{u}+\lambda\mathbf{w}$, with $\lambda>0$ and $\mathbf{w}$ an adapted process in $L^4(\Omega;L^\infty(0,T;\mathbb{H}))\cap L^2(\Omega;L^2(0,T;\mathbb{V}))$. Next we divide by λ and finally we let λ vanish in (3.35) to deduce

$$E\left\{\int_0^T e^{-r(t)}\langle F(\mathbf{u}(t))-F_0(t),\mathbf{w}(t)\rangle\,dt\right\}\geq 0,$$

and because $\mathbf{w}$ is arbitrary, we conclude that $F_0(t)=F(\mathbf{u}(t))$. This proves that $\mathbf{u}$ is a solution of the stochastic Navier–Stokes equation (3.3). □

The technique to identify the limiting drift $F_0(t)$ in (3.35) is a variant of the classic argument used for the monotone operator, see [16] and [18]. The semigroup technique, as in Chapter 15 of [8], provides a pathwise (or mild) solution by means of a stochastic convolution and the change of unknown function $\mathbf{u} := \tilde{\mathbf{u}} + \mathbf{W}_A$, where

$$\mathbf{W}_A(t) := \sum_k \int_0^t \exp[(t-s)A]\mathbf{g}_k(s)\, dw_k(s). \tag{3.36}$$

The (deterministic, with random data) mild equation is as follows:

$$\partial_t \tilde{\mathbf{u}} + A\tilde{\mathbf{u}} + B(\tilde{\mathbf{u}} + \mathbf{W}_A) = \mathbf{f} \qquad \text{in} \quad \mathbb{L}^2(0,T;\mathbb{H}) \cap \mathbb{L}^4(\mathcal{O} \times (0,T)), \tag{3.37}$$

with an initial condition in $\mathbb{H}$. The technique of Proposition 3.3 can be used with the pathwise equation (3.37) to give another proof of the existence of a pathwise (mild) solution.

When the domain $\mathcal{O}$ in $\mathbb{R}^2$ is unbounded, we need to use the norm

$$\|\mathbf{v}\|_{\mathbb{V}} := \left(\int_{\mathcal{O}} |\mathbf{v}|^2 + |\nabla \mathbf{v}|^2\, dx \right)^{1/2}, \tag{3.38}$$

instead of (1.6) for the space $\mathbb{V}$, so that it will be continuously embedded in $\mathbb{H}$. In this case, the a priori estimates (3.17) in Proposition 3.1 and (3.23) remain the same, namely

$$\begin{aligned} E &\left\{ \sup_{0 \le t \le T} |\mathbf{u}^n(t)|^p e^{-\varepsilon t} + \frac{p}{2}\, \nu \int_0^T |\nabla \mathbf{u}^n(t)|^2 |\mathbf{u}^n(t)|^{p-2} e^{-\varepsilon t} dt \right\} \\ &\qquad \le |\mathbf{u}(0)|^p + C_{\varepsilon,p,T,\nu} \int_0^T [\|\mathbf{f}(t)\|_{\mathbb{V}'}^p + \mathrm{Tr}(\mathbf{g}^* \mathbf{g}(t))^{p/2}] e^{-\varepsilon t}\, dt, \end{aligned} \tag{3.39}$$

for some constant $C_{\varepsilon,p,T,\nu}$ depending only on $\varepsilon > 0$, $1 \le p < \infty$, $T > 0$ and $\nu > 0$. Here we need $\varepsilon > 0$ to compensate the seminorm $|\nabla \cdot|$ with the norm (3.38) in the space $\mathbb{V}$. Since we can still control the $\mathbb{L}^4(\mathcal{O})$-norm in terms of the $\mathbb{L}^2(\mathcal{O})$-norms, see estimate (2.2) in Lemma 2.1, the existence and uniqueness results hold for unbounded 2-D domains. On the contrary, in 3-D, estimates in the spaces $L^2(0,T,\mathbb{V})$ and $L^\infty(0,T,\mathbb{H})$ are not enough to ensure a bound in $\mathbb{L}^4(\Omega \times (0,T))$ and the above results are not longer valid.

We may use as initial time τ a stopping time (random variable) with respect to the natural filtration $(\mathcal{F}_t,\ t \ge 0)$ (right-continuous and completed) associated with the Wiener process, and initial value $\mathbf{u}_0 = \mathbf{u}_\tau(x,\omega)$ which is an $\mathcal{F}_\tau$-measurable random variable. Similarly, we may allow random forcing terms $\mathbf{f}(x,t,\omega)$ and $\mathbf{g}(x,t,\omega)$ or even have a smooth dependency on the solution $\mathbf{u}$. For the random initial conditions we have to write the stochastic Navier–Stokes equation (3.3), (3.4) in its integral (variational) form, namely

$$\begin{aligned} (\mathbf{u}(\theta), \mathbf{v}) &+ \int_\tau^\theta \langle A\mathbf{u}(t) + B(\mathbf{u}(t)), \mathbf{v} \rangle\, dt \\ &= (\mathbf{u}_\tau, \mathbf{v}) + \int_\tau^\theta (\mathbf{f}(t), \mathbf{v})\, dt + \sum_k \int_\tau^\theta (\mathbf{g}_k(t), \mathbf{v})\, dw_k(t), \end{aligned} \tag{3.40}$$

for any stopping time $\tau \leq \theta \leq T$ and any $\mathbf{v}$ in the space $\mathbb{V}$. Actually, by a density argument we may allow any adapted process $\mathbf{v}(t)$ in $L^2(\Omega; L^2(\tau, T; \mathbb{V})) \cap \mathbb{L}^4(\mathcal{O} \times (\tau, T) \times \Omega)$. We state the following result valid for smooth domains $\mathcal{O}$ in $\mathbb{R}^2$ not necessarily bounded.

Proposition 3.4 (2-D). *Let τ and $\mathbf{u}_\tau$ be a stopping time with respect to $(\mathcal{F}_t,\ t \geq 0)$ and an $\mathcal{F}_\tau$-measurable random variable such that*

$$0 \leq \tau \leq T, \qquad \mathbf{u}_\tau \in L^4(\Omega; \mathbb{H}). \tag{3.41}$$

Suppose $\mathbf{f}(x, t, \omega)$ and $\mathbf{g}(x, t, \omega)$ are adapted processes such that

$$\mathbf{f} \in L^4((\tau, T) \times \Omega; \mathbb{V}'), \qquad \mathbf{g} \in L^4((\tau, T) \times \Omega; \ell_2(\mathbb{H})). \tag{3.42}$$

Then there exists an adapted process $\mathbf{u}(t, x, \omega)$ with the regularity

$$\mathbf{u} \in L^4(\Omega; C^0(\tau, T; \mathbb{H})) \cap L^2((\tau, T) \times \Omega; \mathbb{V})) \tag{3.43}$$

and satisfying (3.40) *and the following a priori bound holds for $p \geq 2$:*

$$\begin{aligned} &E\left\{\sup_{\tau \leq t \leq T} |\mathbf{u}(t)|^p e^{-\varepsilon t} + \frac{p}{2}\,\nu \int_\tau^T |\nabla \mathbf{u}(t)|^2 |\mathbf{u}(t)|^{p-2} e^{-\varepsilon t}\, dt\right\} \\ &\quad \leq E\left\{|\mathbf{u}(\tau)|^p + C_{\varepsilon,T,\nu} \int_\tau^T [\|\mathbf{f}(t)\|_{\mathbb{V}'}^p + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))^{p/2}] e^{-\varepsilon t}\, dt\right\}, \end{aligned} \tag{3.44}$$

for some constant $C_{\varepsilon,T,\nu}$ depending only on $\varepsilon > 0$, $T > 0$ and $\nu > 0$. Moreover, if $\bar{\mathbf{u}}(t, x, \omega)$ is the solution with another initial data, we have

$$|\mathbf{u}(\theta) - \bar{\mathbf{u}}(\theta)|^2 \exp\left[-\frac{32}{\nu^3} \int_\tau^\theta \|\mathbf{u(t)}\|_{\mathbb{L}^4(\mathcal{O})}^4 dt\right] \leq |\mathbf{u}_\tau - \bar{\mathbf{u}}(\tau)|^2, \tag{3.45}$$

with probability 1, *for any $\tau \leq \theta \leq T$.*

Proof. This is a consequence of previous propositions and the above comments. Notice that we set $\mathbf{u}(t) := \mathbf{u}_\tau$ for any $0 \leq t \leq \tau$. ☐

This proposition is stochastic analogous to the classic results in [14].
Notice that we have

$$\|\mathbf{u}(t)\|_{\mathbb{L}^4(\mathcal{O})}^4 \leq 2|\mathbf{u}(t)|^2\, |\nabla \mathbf{u}(t)|^2, \tag{3.46}$$

so that a priori estimate (3.44) contains the regularity conditions

$$\mathbf{u} \in \mathbb{L}^4(\mathcal{O} \times (0, T) \times \Omega). \tag{3.47}$$

Moreover, the (linear) energy equality (3.7) and estimate

$$\begin{aligned} &E\{|\mathbf{u}(\theta)|^2\} e^{-\varepsilon\theta} + \nu E\left\{\int_\tau^\theta \|\mathbf{u}(t)\|^2 e^{-\varepsilon t}\, dt\right\} \\ &\quad \leq E\{|\mathbf{u}_\tau|^2 e^{-\varepsilon\tau}\} + E\left\{\int_\tau^\theta \left[\frac{1}{\min\{\nu, \varepsilon\}} \|\mathbf{f}(t)\|_{\mathbb{V}'}^2 + \mathrm{Tr}(\mathbf{g}^*\mathbf{g}(t))\right] e^{-\varepsilon t}\, dt\right\} \end{aligned} \tag{3.48}$$

hold. Furthermore, if the domain $\mathcal{O}$ is bounded or forcing term $\mathbf{f}(t)$ is such that for some constant $C = C_{\mathbf{f}}$ we have

$$E\left\{\int_0^T \sup_{|\nabla \mathbf{v}|\le 1} |\langle \mathbf{f}(t), \mathbf{v}\rangle|^4\, dt\right\} \le C_{\mathbf{f}}, \tag{3.49}$$

then we may replace the $\min\{\nu, \varepsilon\}$ with ν in estimate (3.48) and set $\varepsilon = 0$.

For additive noise, a key point used by Bensoussan and Temam [4] and Flandoli and Gatarek [10] is the comparison of the stochastic Navier–Stokes solution (3.3) with the solution of the linear equation

$$dv(t) + Av(t)\, dt = \sum_k \mathbf{g}_k(t)\, dw_k(t),$$

which yields the deterministic Navier–Stokes-type equation

$$\dot{w} + Aw + B(w + v) = f,$$

for the unknown $w = u - v$, and, therefore, the existence of a strong solution can be deduced. However, our technique can also be used with multiplicative noise. Indeed, if the noise takes the form $g(t, u)\, dw(t) = \sum_k \mathbf{g}_k(t, x, u)\, dw_k(t)$, where $g(t, u)$ is a continuous operator from $\mathbb{V}$ into $L^2(0, T; \ell_2(\mathbb{H}))$, we can modify the calculations in the above propositions under the assumption: there is a $\lambda > 0$ such that for some $0 < \nu' < \nu$ we have

$$\sum_k |\mathbf{g}_k(t, u) - \mathbf{g}_k(t, v)|^2_{\mathbb{H}} + \lambda |u - v|^2_{\mathbb{H}} \le \nu' |\nabla u - \nabla v|^2_{\mathbb{H}}, \qquad \forall u, v \in \mathbb{V}.$$

Thus the existence and uniqueness of a strong solution holds even for multiplicative noise.

Remark 3.5 (V-Regularity). It is clear that if the adapted processes $\mathbf{f}$ and $\mathbf{g}$ satisfy

$$\mathbf{f} \in L^2((0, T) \times \Omega; \mathbb{H}), \qquad \mathbf{g} \in L^2((0, T) \times \Omega; \ell_2(\mathbb{V})), \tag{3.50}$$

then the arguments of Remark 2.7 show that the solution of the 2-D stochastic Navier–Stokes equation (3.3) satisfies

$$\mathbf{u} \in C^0(0, T; \mathbb{V}) \cap L^2(0, T; \mathbb{H}^2(\mathcal{O}, \mathbb{R}^2)), \tag{3.51}$$

with probability 1, provided the initial data $\mathbf{u}_\tau$ is in $\mathbb{V}$. More details are needed to obtain an estimate similar to (3.44). Notice that the above assumption (3.50) on the Hilbert–Schmidt operator $\mathbf{g}(t)$ means that $\sum_k \|\mathbf{g_k}(t)\|^2$ is integrable in $(0, T) \times \Omega$. Hence, if $\mathcal{O}$ is bounded, we can follow the arguments in Chapter 15 of [8] to deduce the existence of an invariant measure.

We consider the space $C^0_p(\mathbb{H})$ of real continuous functions h on $\mathbb{H}$ with a p-growth, $0 \le p < \infty$, i.e.,

$$|h(\mathbf{v})| \le C_h(1 + |\mathbf{v}|^p), \qquad \forall\, \mathbf{v} \in \mathbb{H}. \tag{3.52}$$

When $p = 0$, we have all continuous and bounded real functions on the Hilbert space $\mathbb{H}$. We define the (linear) Navier–Stokes semigroup $(\Phi(t, s),\ t \geq s \geq 0)$ as follows:

$$\Phi(t, s)\colon\ C_p^0(\mathbb{H}) \longrightarrow C_p^0(\mathbb{H}), \qquad \Phi(t, s)h(\mathbf{v}) := E\{h(\mathbf{u}(t, s; \mathbf{v}))\}, \tag{3.53}$$

where $\mathbf{u}(t, s; \mathbf{v})$ denotes the solution $\mathbf{u}(x, t, \omega)$ of the stochastic Navier–Stokes equation (3.3) with initial (deterministic) value $\mathbf{u}(x, s, \omega) = \mathbf{v}(x)$. We have

Proposition 3.6 (Markov-Feller). *Suppose we are given two adapted processes* $\mathbf{f}(x, t, \omega)$ *and* $\mathbf{g}(x, t, \omega)$ *satisfying condition* (3.42) *of the previous proposition. Then* $(\Phi(t, s), t \geq s \geq 0)$ *is a Markov–Feller semigroup on the space* $C_p^0(\mathbb{H})$, *for any* $0 \leq p < 6$.

Proof. The uniqueness of solutions yields the semigroup property. Next, by definition we have that $h(\mathbf{v}) \geq 0$ for all $\mathbf{v}$ implies $\Phi(t, s)h(\mathbf{v}) \geq 0$ for all $\mathbf{v}$. Thus we need only check the Feller property and the pointwise convergence at $t = s$, i.e., for any function h in $C_p^0(\mathbb{H})$

$$\begin{cases} \mathbf{v}_n \to \mathbf{v} \text{ in } \mathbb{H} & \text{implies} \quad \Phi(t, s)h(\mathbf{v}_n) \to \Phi(t, s)h(\mathbf{v}), \\ t_n \to s \text{ in } \mathbb{R}^+ & \text{implies} \quad \Phi(t_n, s)h(\mathbf{v}) \to h(\mathbf{v}), \end{cases} \tag{3.54}$$

for any $\mathbf{v}$ in $\mathbb{H}$. Indeed, to show (3.54) we notice that from estimate (3.45) and the continuity of h we deduce that $h(\mathbf{v}_n)$ converges to $h(\mathbf{v})$ in $\mathbb{R}$ with probability 1, and because the solution $\mathbf{u}(t, s; \mathbf{v})$ belongs to $L^2(\Omega; C^0(s, T; \mathbb{H}))$ we have that $h(\mathbf{u}(t_n, s; \mathbf{v}))$ converges to $h(\mathbf{v})$ in $\mathbb{R}$ with probability 1. Hence, the a priori estimate (3.44) lets us take the limits inside the integral for any $p < 4$. □

It is clear that if we need to work in a space $C_p^0(\mathbb{H})$ for some $p \geq 4$ we need to require conditions (3.41) and (3.42) for some $q > p$ instead of just 4.

A realization in the canonical space $C^0(0, T; \mathbb{H})$ of the Markov–Feller process associated with the above semigroup is given by the random field $\mathbf{u}(t, s; \mathbf{v})$, $t > s > 0$, $\mathbf{v}$ in $\mathbb{H}$, i.e., the solution of stochastic PDE (3.40) with initial value $\tau = s$ and $\mathbf{v}_\tau = \mathbf{v}$.

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Accepted 5 *May* 2002. *Online publication* 9 *September* 2002.