Bellman Equations Associated to the Optimal Feedback
Control of Stochastic Navier-Stokes Equations

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Abstract
In this paper we study infinite-dimensional, second-order Hamilton-Jacobi-Bellman equations associated to the feedback synthesis of stochastic Navier-Stokes equations forced by space-time white noise. Uniqueness and existence of viscosity solutions are proven for these infinite-dimensional partial differential equations. © 2004 Wiley Periodicals, Inc.

1 Introduction
Feedback control theory of fluid mechanics has been an emerging subject in applied mathematics with potential utility in several important engineering applications [14, 21]. One of the outstanding problems in this subject is the rigorous resolution of the feedback synthesis of optimal control problems for stochastic Navier-Stokes equations using infinite-dimensional Hamilton-Jacobi-Bellman (HJB) equations. In [12] first-order HJB equations associated to the control of deterministic Navier-Stokes equations were considered and existence and uniqueness of viscosity solutions were proven.

In this paper we study the infinite-dimensional, second-order HJB equations that arise in problems of optimal control of stochastic Navier-Stokes equations. There is very little known about equations of this type. Kolmogorov equations for stochastic Navier-Stokes equations have been studied by Komech and Vishik (see [23] and the references therein) and more recently by Flandoli and Gozzi [9] for the two-dimensional stochastic Navier-Stokes equations, and by Da Prato and Debussche [6] for the three-dimensional case. Only existence of strict and mild solutions has been proven in [6]. A semilinear equation associated to a special optimal control problem has been investigated by Da Prato and Debussche in [5] from the point of view of mild solutions using an exponential change of variables.
that reduced the equation to a more treatable one. We approach the problem from the point of view of viscosity solutions. We only consider semilinear equations; however, the theory could be extended very easily to fully nonlinear equations by standard existing techniques. Our motivating example is the HJB equation that comes from the problem of minimizing the enstrophy (the square integral of the vorticity field) for the stochastic flow. (This is also the case formulated in [21] and studied in [5].) We introduce a different definition of viscosity solution compared to the one used in [12]. This definition allows us to handle cost functionals that are very singular. Moreover, it also applies to the first-order HJB equations investigated in [12] and can thus lead to an alternative theory of these equations that covers some cases not treatable by the techniques of [12].

The equations considered in this paper are more general than those investigated in [5]. We can handle more complicated cost functionals and apply our approach to stochastic optimal control problems with the associated HJB equations that are fully nonlinear in the gradient variable. Regarding the nondegeneracy, the equations covered by our paper and those of [5] are fairly complementary. In [5] the Wiener process in the stochastic Navier-Stokes equations must be nondegenerate and must satisfy additional conditions that allow the associated Ornstein-Uhlenbeck semigroup to have smoothing properties. We do not need such assumptions here. The noise can be totally degenerate, i.e., the covariance operator $Q$ (see Section 2) can be degenerate, even 0. However, we have to assume condition (2.7), which precludes us from treating nondegenerate cases of [5]. Moreover, since we deal with nonsmooth solutions, we do not derive a formula for optimal feedback as was done in [5], and it is not clear at this time if the results of this paper can be extended to the case of stochastic Navier-Stokes equations with nonperiodic boundary conditions. This is because we use property (2.3) of the bilinear form $b$ defined in Section 2, which is only true in two dimensions for periodic boundary conditions. We think that a good theory for the Dirichlet boundary condition is possible, and we plan to come back to the nonperiodic case in a future publication.

We will consider an optimal control problem for the two-dimensional stochastic Navier-Stokes equations with periodic boundary conditions. The controlled stochastic Navier-Stokes equations can be rewritten as an abstract stochastic evolution equation for the velocity vector field (see, e.g., [23], [21, chap. I], and [22] for more on this.) We will deal with this abstract form, which we describe below.

Let $D = [0, L] \times [0, L]$, and let $\nu > 0$. We define the spaces

$$
\mathbf{H} = \text{the closure of } \{ x \in H^1(D; \mathbb{R}^2), \text{div} \, x = 0, \int_D x = 0 \} \text{ in } L^2(D; \mathbb{R}^2),
$$

$$
\mathbf{V} = \{ x \in H^1(D; \mathbb{R}^2), \text{div} \, x = 0, \int_D x = 0 \},
$$

where for an integer $k \geq 1$, $H^k(D; \mathbb{R}^2)$ is the space of $\mathbb{R}^2$-valued functions $x$ that are in $H^k_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and such that $x(y + Le_i) = x(y)$ for every $y \in \mathbb{R}^2$ and $i = 1, 2$. 
We recall here that the enstrophy

\begin{equation}
\int_D |\text{curl} \, \mathbf{x}(\xi)|^2 \, d\xi = \int_D |\nabla \mathbf{x}(\xi)|^2 \, d\xi \quad \text{for } \mathbf{x} \in \mathbf{V}.
\end{equation}

Hence the \( \|\mathbf{x}\|_V \)-norm is equivalent to

\begin{equation}
\left( \int_D |\text{curl} \, \mathbf{x}(\xi)|^2 \, d\xi \right)^{1/2}.
\end{equation}

Let \( \mathbf{P}_H \) be the orthogonal projection in \( L^2(D; \mathbb{R}^2) \) onto \( \mathbf{H} \). Define \( \mathbf{A} \mathbf{x} = -\mathbf{P}_H \Delta \mathbf{x} \) and \( \mathbf{B}(\mathbf{x}, \mathbf{y}) = \mathbf{P}_H[(\mathbf{x} \cdot \nabla)\mathbf{y}] \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, and let \( \mathbf{W} \) be an \( \mathbf{H} \)-valued Wiener process on this probability space with the covariance \( \mathbf{P}_H \). Given an initial time \( t \geq 0 \) and terminal time \( T \geq t \), the abstract controlled stochastic Navier-Stokes equations describe the evolution of the velocity vector field \( \mathbf{X} : [t, T] \times D \times \Omega \rightarrow \mathbb{R}^2 \) that satisfies the Itô-type equation

\begin{equation}
\begin{cases}
\frac{d\mathbf{X}(s)}{ds} = (-v \mathbf{A} \mathbf{X}(s) - \mathbf{B}(\mathbf{X}(s), \mathbf{X}(s))) \\
\quad + f(s, a(s)) \, ds + d\mathbf{W}(s) \quad \text{in } (t, T] \times \mathbf{H},
\end{cases}
\end{equation}

where \( f : [0, T] \times \mathbb{U} \rightarrow \mathbb{V}, \mathbb{U} \) is a complete, separable metric space, and \( a(\cdot) : [0, T] \times \Omega \rightarrow \mathbb{U} \) is a stochastic process that plays the role of a control strategy. Given the initial time \( t \), the set of control strategies will be denoted by \( \mathcal{U}_t \). We will describe it precisely in the next section.

The optimal control problem consists in the minimization, over all controls \( a(\cdot) \in \mathcal{U}_t \), of a cost functional

\[ J(t, \mathbf{x}; a(\cdot)) = \mathbb{E} \left\{ \int_t^T l(s, \mathbf{X}(s), a(s)) \, ds + g(\mathbf{X}(T)) \right\}. \]

The dynamic programming approach to the control problem involves the study of the value function

\[ \mathcal{V}(t, \mathbf{x}) = \inf_{a(\cdot) \in \mathcal{U}_t} J(t, \mathbf{x}; a(\cdot)) \]

and its goal is the characterization of \( \mathcal{V} \) as a solution of the associated Hamilton-Jacobi-Bellman partial differential equation

\begin{equation}
\begin{cases}
u_t + \frac{1}{2} \text{tr}(Q D^2 u) - \langle v \mathbf{A} \mathbf{x} + \mathbf{B}(\mathbf{x}, \mathbf{x}), Du \rangle \\
\quad \quad + \inf_{a \in \mathbb{U}} \{ f(t, \mathbf{a}, Du) + l(t, \mathbf{x}, \mathbf{a}) \} = 0 \quad \text{for } (t, \mathbf{x}) \in (0, T) \times \mathbf{H},
\end{cases}
\end{equation}

Here the Hamiltonian function is denoted as

\begin{equation}
F(t, \mathbf{x}, \mathbf{p}) := \inf_{a \in \mathbb{U}} \{ f(t, \mathbf{a}, \mathbf{p}) + l(t, \mathbf{x}, \mathbf{a}) \}. \end{equation}
The idea then is to use the HJB equation to construct optimal feedback controls, obtain verification theorems, and do numerical computations. This program has not yet been carried out in infinite dimensions. Theoretical results on (1.4) are its first step.

We will study a more general class of infinite-dimensional HJB equations

\[
\begin{cases}
    u_t + \frac{1}{2} \text{tr}(Q D^2 u) \\
    - \langle v Ax + B(x, x), Du \rangle + F(t, x, Du) + H(t, x) = 0 \quad \text{for} \ (t, x) \in (0, T) \times H, \\
    u(T, x) = g(x), \quad x \in H.
\end{cases}
\]

We will introduce an appropriate notion of viscosity solution and prove a comparison theorem that applies to (1.6). Moreover, we will prove the existence of viscosity solutions for (1.4). The proof of existence of viscosity solutions will be based on stochastic optimal control techniques. This is why the full theory we present here is in some respect limited to equation (1.4).

The following specific example is covered by the analysis of this paper (Hypothesis 6.1 in Section 6). Let

\[
\begin{align*}
    l(t, x, a) &= \| \text{curl} x \|^2 + \frac{1}{2} \| a \|^2, \\
    g(x) &= \| x \|^2, \\
    f(t, a) &= Ka \in V,
\end{align*}
\]

with \( K \in \mathcal{L}(H; V) \) and \( U = B_H(0, R) \subset H \) being the ball of radius \( R \) in \( H \). Such a control and the singular kernel of \( K \) can be approximately realized by a suitable Lorentz force distribution in electrically conducting fluids such as liquid metals and salt water. The Hamiltonian function is then

\[
F(t, x, p) = \| \text{curl} x \|^2 + h(K^* p),
\]

where \( h(\cdot) : H \to R \) is given by

\[
h(z) := \inf_{a \in U} \left\{ \langle a, z \rangle_H + \frac{1}{2} \| a \|^2 \right\}.
\]

We can, in fact, explicitly obtain \( h(\cdot) \) as

\[
h(z) = \begin{cases} 
-\frac{1}{2} \| z \|^2 & \text{for } \| z \| \leq R \\
-\| z \| + \frac{1}{2} R^2 & \text{for } \| z \| > R.
\end{cases}
\]

Moreover, the optimal feedback control is given formally as

\[
\tilde{a}(t) = \Upsilon(K^* D_x u(t, x(t))),
\]

where the function \( \Upsilon(\cdot) \) is given by

\[
\Upsilon(z) := D_z h(z) = \begin{cases} 
-z & \text{for } \| z \| \leq R \\
-z \frac{R}{\| z \|} & \text{for } \| z \| > R.
\end{cases}
\]
2 Preliminaries

The operator

\[ A = -P_H \Delta \]

that appeared in the introduction has the domain

\[ D(A) = H^2(D; \mathbb{R}^2) \cap V. \]

It is positive definite and self-adjoint, and \( A^{-1} \) is compact. We will denote by \( \langle \cdot, \cdot \rangle \), and \( \| \cdot \| \), respectively, the inner product and the norm in \( L^2(D; \mathbb{R}^2) \). The space \( H \) inherits the same inner product and norm.

For \( \gamma = 1, 2 \) we denote by \( V_\gamma \) the domain of \( A^{\gamma/2} \), \( D(A^{\gamma/2}) \), equipped with the norm

\[ \| x \|_\gamma = \| A^{\gamma/2} x \|. \]

It is well known that the space \( V_1 \) coincides with \( V \).

It is convenient to introduce the trilinear form \( b(\cdot, \cdot, \cdot) : V \times V \times V \rightarrow \mathbb{R} \). It is defined as

\[ b(x, y, z) = \int_{\Omega} z(\xi) \cdot (x(\xi) \cdot \nabla_\xi) y(\xi) \, d\xi = \langle B(x, y), z \rangle. \]

It is a continuous operator on \( V \times V \times V \), but it can also be extended to a continuous map in different topologies; for instance, it is also continuous on \( V \times V_2 \times H \) (see [22] and (2.6) below). The incompressibility condition gives the standard orthogonality relations

\[ b(x, y, z) = -b(x, z, y), \quad b(x, y, y) = 0. \]

Also, because of the periodic boundary conditions (see, e.g., [22])

\[ b(x, x, Ax) = 0 \quad \text{for} \ x \in V_2. \]

We will be using the following inequalities: If \( x, y, z \in V \), then

\[ |b(x, y, z)| \leq C \| x \|^{1/2} \| x \|_1^{1/2} \| y \|_1 \| z \|^{1/2} \| z \|_1^{1/2}, \]

which gives when \( z = x \)

\[ |b(x, y, x)| \leq C \| x \| \| x \|_1 \| y \|_1. \]

Also, if \( x \in V, y \in V_2, \) and \( z \in H \), then

\[ |b(x, y, z)| \leq C \| x \|_1 \| y \|_2 \| z \|. \]

The covariance operator \( Q : H \rightarrow H \) is self-adjoint, \( Q \geq 0 \), and

\[ \text{tr}(Q) < +\infty. \]

We point out that \( Q \) can be totally degenerate, contrary to the case of [5]. Denote \( Q_1 = A^{1/2} Q A^{1/2}. \) We will require throughout the paper that

\[ \text{tr}(Q_1) < +\infty. \]
We also assume throughout the paper that $U$ is a complete, separable metric space.

We will work with the canonical sample space for the controlled stochastic Navier-Stokes equations. For $0 \leq t \leq T$, let $\Omega_t = \{ \omega \in C([t, T]; H) : \omega(t) = 0 \}$. The Wiener process $W$ is defined on $\Omega_t$ by $W(\tau)(\omega) = \omega(\tau)$. Let $F_{t,s}$ be the $\sigma$-algebra generated by paths of $W$ up to time $s$ in $\Omega_t$, and let $P_t$ be the Wiener measure on $\Omega_t$ (see [7, 17]). Then $(\Omega_t, F_{t,T}, F_{t,s}, P_t)$ is the canonical sample space for the Wiener process $W$.

We say that $a(\cdot) : [t, T] \times \Omega_t \to U$ is an admissible control on $[t, T]$ if $a(\cdot)$ is an $F_{t,s}$-progressively-measurable process. The set of all admissible controls on $[t, T]$ will be denoted by $U_t$.

Finally, for $0 \leq t \leq T$ and a real separable Hilbert space $Z$, we will denote by $M^2(t, T; Z)$ the space of all $F_{t,s}$-progressively-measurable processes $z : [t, T] \times \Omega_t \to Z$ such that

$$\mathbb{E}\left( \int_t^T \|z(s)\|^2_Z ds \right) < +\infty.$$

**Remark 2.1.** Our definition of admissible controls is different from the most common one, in which the probability spaces are allowed to vary (see, e.g., [24, chap. 5, sec. 4]). We have chosen to work with the canonical sample space because we want to present a self-contained proof of the dynamic programming principle that avoids the general issue of weak uniqueness for the controlled stochastic Navier-Stokes equations.

## 3 Controlled Stochastic Navier-Stokes Equation: Existence, Uniqueness, and Continuous Dependence

In this section we derive various estimates for solutions of the stochastic Navier-Stokes equation (1.3). We first recall the definition of a solution (see, e.g., [7, 19, 23]).

**Definition 3.1**

(i) A variational solution of equation (1.3) is a process $X \in M^2(t, T; V)$ such that for every $z \in V$ and every $s \in (t, T)$, we have

$$\langle X(s), z \rangle = \langle x, z \rangle + \int_t^s \left\langle -AX(\tau) - B(X(\tau), X(\tau)) + f(\tau, a(\tau)), z \right\rangle d\tau$$

$$+ \int_t^s \langle z, dW(\tau) \rangle.$$

(ii) A strong solution of equation (1.3) is a process

$X \in M^2(t, T; V_2)$

such that for every $s \in (t, T)$, we have

$$X(s) = x + \int_t^s ( -AX(\tau) - B(X(\tau), X(\tau)) + f(\tau, a(\tau))) d\tau + \int_t^s dW(\tau).$$
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(The first integral above is the Bochner integral.)

In the above \( \int_t^s dW(\tau) = W(s) \). We note here that both of the above definitions refer to “strong solutions” in the language of stochastic analysis. On the other hand, the notion of weak solutions in this context refers to the martingale solutions, where the probability space is obtained as part of the solution.

The next two propositions collect existence, uniqueness, and continuous dependence results for variational and strong solutions of the stochastic Navier-Stokes equation (1.3).

**Proposition 3.2** Let \( f : [0, T] \times U \to V \) be continuous and such that

\[
\|f(t, a)\|_1 \leq R \quad \text{for all } t \in [0, T], a \in U.
\]

Let \( 0 \leq t \leq T \), and let \( a(\cdot) \in \mathcal{L}_u \).

(i) If \( \mathbb{E}_t \|x\|^p < +\infty, p \geq 2 \), then there exists a unique variational solution \( X(\cdot) = X(\cdot ; t, x, a(\cdot)) \) of the state equation (1.3). Moreover, we have for \( t \leq s \leq T \)

\[
\mathbb{E}_t \left\| X(s) \right\|^p + v \mathbb{E}_t \int_t^s \left\| X(\tau) \right\|^2 \left\| X(\tau) \right\|^{p-2} d\tau \leq \mathbb{E}_t \left\| x \right\|^p + C_{v,p}(R^p + (\text{tr}(Q))^{p/2})(s-t).
\]

(ii) If \( \mathbb{E}_t \|x\|_1^p < +\infty, p \geq 2 \), then there exists a unique strong solution \( X(\cdot) = X(\cdot ; t, x, a(\cdot)) \) of the state equation (1.3). Moreover, we have for \( t \leq s \leq T \)

\[
\mathbb{E}_t \left\| X(s) \right\|^p + v \mathbb{E}_t \int_t^s \left\| X(\tau) \right\|^2 \left\| X(\tau) \right\|_1^{p-2} d\tau \leq \mathbb{E}_t \left\| x \right\|^p + C_{v,p} \left( R^p + (\text{tr}(Q_1))^{p/2} \right)(s-t)
\]

and

\[
\sup_{t \leq s \leq T} \mathbb{E}_t \left\| X(s) \right\|^p_1 \leq C(v, p, T, R, \text{tr}(Q_1))(1 + \mathbb{E}_t \|x\|^p)
\]

for some constant \( C(v, p, T, R, \text{tr}(Q_1)) \).

**Proof:** The existence and uniqueness of variational and strong solutions is standard [19, 23]. The energy inequality (3.1) can be found, for instance, in [19], and estimates (3.2) and (3.3) can be proven similarly applying Itô’s formula [7, 15, 17] to the function \( \| \cdot \|_1^p \) and a standard martingale inequality (see, e.g., [7, p. 78]). □

**Proposition 3.3** Let \( f : [0, T] \times U \to V \) be as in Proposition 3.2, let \( p \geq 2 \), and let \( t < s \leq T \). Then we have the following:
(i) There exists a constant $C_\nu$ such that for every $a(\cdot) \in \mathcal{U}_\nu$, and all initial conditions $x, y \in H$

\begin{equation}
\|X(s) - Y(s)\|^2 + v \int_t^s \|X(\tau) - Y(\tau)\|^2 \, d\tau \leq \|x - y\|^2 \exp \left\{ \int_t^s C_\nu \|Y(\tau)\|^2 \, d\tau \right\}
\end{equation}

a.s. on $\Omega_\nu$, where $X(\cdot) = X(\cdot; t, x, a(\cdot))$ and $Y(\cdot) = Y(\cdot; t, y, a(\cdot))$ are solutions of (1.3) with initial conditions $x$ and $y$, respectively.

(ii) There exists a constant $C = C(p, v, R, \|x\|_1, \text{tr}(Q))$ such that for every $a(\cdot) \in \mathcal{U}_\nu$ and every initial condition $x \in V$

\begin{equation}
\mathbb{E}_t \|X(s) - x\|^p \leq C(p, v, R, \|x\|_1, \text{tr}(Q))(s - t),
\end{equation}

where $X(\cdot) = X(\cdot; t, x, a(\cdot))$.

(iii) For every initial condition $x \in V$ there exists a modulus $\omega$ independent of the strategies $a(\cdot) \in \mathcal{U}_\nu$ such that

\begin{equation}
\mathbb{E}_t \|X(s) - x\|_1^2 \leq \omega(s - t),
\end{equation}

where $X(\cdot) = X(\cdot; t, x, a(\cdot))$.

**Proof of (i):** Denote $Z(s) = X(s) - Y(s)$. Then $Z(s)$ satisfies

\begin{equation*}
Z(s) = x - y - v \int_t^s A Z(\tau) \, d\tau + \int_t^s [B(Y(\tau)) - B(X(\tau))] \, d\tau.
\end{equation*}

Hence using (2.2) and (2.5), we obtain

\begin{equation}
\|Z(s)\|^2 = \|x - y\|^2 - 2v \int_t^s \|Z(\tau)\|_1^2 \, d\tau + \int_t^s b(Z(\tau), X(\tau), Z(\tau)) \, d\tau
\leq \|x - y\|^2 - 2v \int_t^s \|Z(\tau)\|_1^2 \, d\tau + \int_t^s C \|Z(\tau)\|_1 \|X(\tau)\|_1 \|Z(\tau)\|_2 \, d\tau
\leq \|x - y\|^2 - 2v \int_t^s \|Z(\tau)\|_1^2 \, d\tau + C_\nu \int_t^s \|X(\tau)\|_2^2 \|Z(\tau)\|_2^2 \, d\tau.
\end{equation}

Here we have used Young’s inequality. Then it follows from Gronwall’s inequality that

\begin{equation*}
\|Z(s)\|^2 \leq \|x - y\|^2 \exp \left\{ \int_t^s C_\nu \|X(\tau)\|_2^2 \, d\tau \right\} \quad \mathbb{P}_t \text{ a.s.}
\end{equation*}

Plugging this back into (3.7) yields (3.4) with another constant $C_\nu$. \qed

**Proof of (ii):** Denote $Y(s) = X(s) - x$. Then

\begin{equation*}
Y(s) = \int_t^s (-vAX(\tau) - B(X(\tau), X(\tau)) + f(\tau, a(\tau))) \, d\tau + \int_t^s dW(\tau).
\end{equation*}
Therefore, applying Itô’s formula, taking expectations, and using (2.2) and the Cauchy-Schwartz inequality, we obtain
\[
\mathbb{E}_t \| Y(s) \|^p \\
\leq \mathbb{E}_t \int_t^s p \left\{-vAX(\tau) - B(X(\tau), X(\tau)) + f(\tau, a(\tau), Y(\tau))\right\} \| Y(\tau) \|^{p-2} d\tau \\
+ \mathbb{E}_t \int_t^s \frac{p(p - 1)}{2} \text{tr}(Q) \| Y(\tau) \|^{p-2} d\tau \\
\leq - \frac{pv}{2} \mathbb{E}_t \int_t^s \| X(\tau) \|^2 \| Y(\tau) \|^{p-2} d\tau + Cp, v \mathbb{E}_t \int_t^s \| X(\tau) \|^2 \| Y(\tau) \|^{p-2} d\tau \\
+ C(p, v, R, \| x \|_1, \text{tr}(Q)) \mathbb{E}_t \int_t^s (\| Y(\tau) \|^{p-1} + \| Y(\tau) \|^{p-2}) d\tau \\
+ p \mathbb{E}_t \int_t^s |b(X(\tau), X(\tau), x)| \| Y(\tau) \|^{p-2} d\tau.
\]
We now estimate
\[
|b(X(\tau), X(\tau), x)| \leq C \| X(\tau) \|_1 \| X(\tau) \| \| x \|_1
\leq \frac{pv}{2} \| X(\tau) \|^2 + C(p, v) \| X(\tau) \| \| x \|_1^2.
\]
Plugging this into the previous inequality and using (3.1) finally yields
\[
\mathbb{E}_t \| Y(s) \|^p \\
\leq C(p, v, \| x \|_1, \text{tr}(Q)) \mathbb{E}_t \int_t^s (\| Y(\tau) \|^{p-1} + \| Y(\tau) \|^{p-2}) d\tau \\
+ \| X(\tau) \|^2 \| Y(\tau) \|^{p-2} d\tau
\]
\[
\leq C(p, v, \| x \|_1, \text{tr}(Q))(s - t).
\]

\(\square\)

**Proof of (iii):** We want to show that
\[
\sup_{s \in [t, t+\epsilon]} \mathbb{E}_t \| X(s; t, x, a(\cdot)) - x \|_1^2 \to 0 \quad \text{as} \quad \epsilon \to 0.
\]
If not, then there exist sequences \(s_n\) and \(a_n(\cdot)\) such that
\[
s_n \to t, \quad \mathbb{E}_t \| X(s_n; t, x, a_n(\cdot)) - x \|_1^2 \geq \epsilon.
\]
However, by (3.5) and (3.2), we have
\[
X(s_n; t, x, a_n) \to x \quad \text{strongly in} \quad L^2(\Omega_t, H) \quad \text{and weakly in} \quad L^2(\Omega_t, V).
\]
The weak convergence in \(L^2(\Omega_t, V)\) implies that
\[
\| x \|_1^2 \leq \liminf_{n \to \infty} \mathbb{E}_t \| X(s_n; t, x, a_n(\cdot)) \|_1^2.
\]
and from (3.2) we obtain
\[ \|x\|_1^2 \geq \limsup_{n \to \infty} \mathbb{E}_t \|X(s_n; t, x, a_n(\cdot))\|_1^2. \]
These two inequalities yield the convergence of the \( L^2(\Omega_t, V) \) norms and this, together with the weak convergence in \( L^2(\Omega_t, V) \), gives that \( X(s_n; t, x, a_n(\cdot)) \to x \) strongly in \( L^2(\Omega_t, V) \), contrary to our assumption. \( \square \)

4 Viscosity Solutions of the HJB Equation

The definition of viscosity solution we propose here borrows some ideas from \[1, 4, 16\]. Ishii \[16\] used a convex function (that was defined only on a proper subspace) as part of the test functions to deal with unboundedness in the equation. Ishii’s definition has been successfully used in \[20\] to treat some equations that may come from the control of deterministic Navier-Stokes equations. A similar idea based on the use of energy functions also appeared recently in \[8\]. Crandall and Lions \[4\] and Cannarsa and Tessitore \[1\] used special radial functions and the coercivity of the unbounded operators in the state equations to improve the regularity of points where maxima and minima occur in the definition of viscosity solution. This idea has been successfully adapted to second-order equations in \[11, 13\] and also in \[12\].

Here we merge these two approaches. By using a special radial function of \( \|x\|_1 \) we first make the points where maxima or minima occur in the definition of the solution to be in \( V \). Then we are able to show that if the function tested is the value function, properties of the Navier-Stokes equation and the coercivity of the operator \( A \) force these points to be in \( V_2 \); this is what we require as part of the definition. Having this property we can very easily interpret all terms appearing in the HJB equation.

**Definition 4.1** A function \( \psi \) is a test function for equation (1.6) if \( \psi = \varphi + \delta(1 + \|x\|_1^2)^m \), where
- \( \varphi \in C^{1,2}((0, T) \times H) \) and is such that \( \varphi, \varphi_t, D\varphi, \) and \( D^2\varphi \) are uniformly continuous on \( [\epsilon, T - \epsilon] \times H \) for every \( \epsilon > 0 \) and
- \( \delta \in C^1((0, T)) \) is such that \( \delta > 0 \) on \( (0, T) \) and \( m \geq 1 \).

The function \( h(x) = (1 + \|x\|_1^2)^m \) is not Fréchet-differentiable in \( H \). Therefore the terms \( \langle Ax + B(x, x), Dh(x) \rangle \) and \( \text{tr}(Q D^2h(x)) \) have to be understood properly. From the point of view of the HJB equation, it would be best to set it up in the space \( (0, T) \times V \). However, because of the associated control problem, we want to keep \( H \) as our reference space. We define
\[
Dh(x) = 2m(1 + \|x\|_1^2)^{m-1}Ax,
\]
\[
D^2h(x) = 2m(1 + \|x\|_1^2)^{m-1}A + 4m(m - 1)(1 + \|x\|_1^2)^{m-2}Ax \otimes Ax,
\]
and in what follows we will write
\[
D\psi = D\varphi + Dh, \quad D^2\psi = D^2\varphi + D^2h,
\]
even though this is a slight abuse of notation since, as we mentioned before, \( h \) is not Fréchet-differentiable in \( H \). Since, as we will soon see, we require the maximum/minimum point \( x \) in the definition of viscosity solution to be in \( V_2 \), we can rewrite everything using the above defined \( Dh \) and \( D^2h \) and using the inner product and the trace in \( H \). Perhaps this will be clearer in Section 5, where the forms of the expressions involving \( h \) will appear as direct consequences of Itô’s formula applied to \( h \).

We assume that \( F : [0, T] \times V \times H \to \mathbb{R} \).

**Definition 4.2** A function \( u : (0, T) \times V \to \mathbb{R} \) that is weakly sequentially upper-semicontinuous (respectively, lower-semicontinuous) on \((0, T) \times V\) is called a viscosity subsolution (respectively, viscosity supersolution) of (1.6) if, for every test function \( \psi \), whenever \( u - \psi \) has a global maximum (respectively, \( u + \psi \) has a global minimum) over \((0, T) \times V\) at \((t, x)\), then \( x \in V_2 \) and

\[
\psi_t(t, x) + \frac{1}{2} \text{tr}(QD^2\psi(t, x)) - \langle vAx + B(x, x), D\psi(t, x) \rangle + F(t, x, D\psi(t, x)) \geq 0
\]

(respectively,

\[
-\psi_t(t, x) - \frac{1}{2} \text{tr}(QD^2\psi(t, x)) + \langle vAx + B(x, x), D\psi(t, x) \rangle + F(t, x, -D\psi(t, x)) \leq 0.
\]

A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

We remark that the definition of viscosity solution given above is meaningful only for functions with at most polynomial growth at infinity.

## 5 Comparison Principle

In this section we prove a comparison theorem for equation (1.6).

**Hypothesis 5.1.** \( F : [0, T] \times V \times H \to \mathbb{R} \), and there exist a modulus of continuity \( \omega \) and moduli \( \omega_r \) for \( r > 0 \) such that

\[
|F(t, x, p) - F(t, y, p)| \leq \omega_r(||x - y||_1) + \omega(||x - y||_1||p||) \quad \text{if } ||x||_1, ||y||_1 \leq r,
\]

\[
|F(t, x, p) - F(t, x, q)| \leq \omega((1 + ||x||_1)||p - q||),
\]

\[
|F(t, x, p) - F(s, x, p)| \leq \omega_r(|t - s|) \quad \text{if } ||x||_1, ||y||_1, ||p||_1 \leq r.
\]

\[
|g(x) - g(y)| \leq \omega_r(||x - y||) \quad \text{if } ||x||_1, ||y||_1 \leq r.
\]

**Theorem 5.2** Let Hypothesis 5.1 hold. Let \( u, v : (0, T) \times V \to \mathbb{R} \) be, respectively, a viscosity subsolution and a viscosity supersolution of (1.6). Let

\[
u(t, x), -v(t, x), |g(x)| \leq C(1 + ||x||_1^k)
\]
for some \( k > 0, \) and let
\[
(i) \quad \lim_{t \to T} (u(t, x) - g(x))^+ = 0,
(ii) \quad \lim_{t \to T} (v(t, x) - g(x))^- = 0,
\]
uniformly on bounded subsets of \( V. \) Then \( u \leq v \) on \((0, T] \times V.\)

**Proof:** Given \( \mu > 0, \) define
\[
u_{\mu}(t, x) = u(t, x) - \frac{\mu}{t}, \quad v_{\mu}(t, x) = v(t, x) + \frac{\mu}{t}.
\]
Then \( u_{\mu} \) and \( v_{\mu} \) satisfy, respectively,
\[
(u_{\mu},)_t + \frac{1}{2} \text{tr}(QD^2u_{\mu}) - (vAx + B(x, x), Du_{\mu}) + F(t, x, Du_{\mu}) \geq \frac{\mu}{T^2}
\]
and
\[
(v_{\mu},)_t + \frac{1}{2} \text{tr}(QD^2v_{\mu}) - (vAx + B(x, x), Dv_{\mu}) + F(t, x, Dv_{\mu}) \leq -\frac{\mu}{T^2}.
\]
Let \( m \) be a number such that \( m \geq 1 \) and \( 2m \geq k + 1. \) For \( \epsilon, \delta, \gamma > 0 \) and \( 0 < T_\delta < T, \) we consider the function
\[
\Phi(t, s, x, y) = u_{\mu}(t, x) - v_{\mu}(s, y) - \frac{\|x - y\|^2}{2\epsilon} - \delta e^{K_{\mu}(T-t)}(1 + \|x\|_1^2)^m
\]
\[
- \delta e^{K_{\mu}(T-s)}(1 + \|y\|_1^2)^m - \frac{(t - s)^2}{2\gamma}
\]
and set
\[
\Phi(t, s, x, y) = -\infty \quad \text{if } x, y \not\in V.
\]
The constant \( K_{\mu} \) will be chosen later. We claim that this function is weakly sequentially upper-semicontinuous on \((0, T] \times H.\)

It is well known that functions \( x \to (1 + \|x\|_1^2)^m, \) \( y \to (1 + \|y\|_1^2)^m, \) and \( \|x - y\|^2 \)
are weakly sequentially lower-semicontinuous. To show that, say,
\[
u_{\mu}(t, x) - \delta e^{K_{\mu}(T-t)}(1 + \|x\|_1^2)^m
\]
is weakly sequentially upper-semicontinuous on \((0, T] \times H, \) we suppose that this is not the case; i.e., there exist sequences \( t_n \to t, x_n \) and \( t, x \) such that \( t_n \to t, \)
\( x_n \to x \in H, \) and such that
\[
\lim_{n \to \infty} \sup_{t_n} \left( u_{\mu}(t_n, x_n) - \delta e^{K_{\mu}(T-t_n)}(1 + \|x_n\|_1^2)^m \right) >
\]
\[
\lim_{n \to \infty} \left( u_{\mu}(t, x) - \delta e^{K_{\mu}(T-t)}(1 + \|x\|_1^2)^m \right).
\]
If \( \lim_{n \to \infty} \|x_n\|_1 = +\infty, \) this is impossible by (5.5). So there must exist a subsequence (still denoted by \((t_n, x_n)\)) such that \( \limsup_{n \to \infty} \|x_n\|_1 < +\infty. \) But then we have \( x_n \to x \in V, \) which contradicts the weak sequential upper-semicontinuity of \( u_{\mu}. \)

Therefore \( \Phi \) has a global maximum over \((0, T_\delta] \times V \) at some points \( \bar{t}, \bar{s}, \bar{x}, \) and \( \bar{y}, \) where \( 0 < \bar{t}, \bar{s} \) and \( \bar{x} \) and \( \bar{y} \) are bounded independently of \( \epsilon \) for a fixed \( \delta. \) We can
assume this maximum to be strict. Moreover, \( \hat{x}, \hat{y} \in V_2 \). It is standard to notice (see, e.g., [16]) that

\[
\limsup_{\epsilon \to 0} \limsup_{\gamma \to 0} \frac{\|\hat{x} - \hat{y}\|^2}{2\epsilon} = 0 \text{ for fixed } \delta
\]

and

\[
\limsup_{\gamma \to 0} \frac{(\hat{t} - \hat{s})^2}{2\gamma} = 0 \text{ for fixed } \delta, \epsilon.
\]

If \( u \not\subseteq v \), it then follows from (5.7), (5.8), (5.4), and (5.6) that for small \( \mu \) and \( \delta \) and for \( T_\delta \) sufficiently close to \( T \) we have \( \hat{t}, \hat{s} < T_\delta \) if \( \gamma \) and \( \epsilon \) are sufficiently small.

We will now use a rather standard technique of reduction to finite-dimensional spaces to produce appropriate test functions. For a while our presentation will follow rather closely such a reduction described in [11]. (A finite-dimensional reduction technique was first introduced in [18].)

Let \( H_1 \subseteq H_2 \subseteq \cdots \) be finite-dimensional subspaces of \( H \) generated by eigenvectors of \( A \) such that \( \bigcup_{N=1}^\infty H_N = H \). Given \( N \in \mathbb{N}, N > 1 \), denote by \( P_N \) the orthogonal projection onto \( H_N \), let \( Q_N = I - P_N \), and \( H_N^\perp = Q_N H \). (Remember that \( Q_1 \) was defined in Section 2 and is not a projection.) We then have an orthogonal decomposition \( H = H_N \times H_N^\perp \), and we will denote by \( x_N \) an element of \( H_N \) and by \( x_N^\perp \) an element of \( H_N^\perp \). For \( x \in H \), we will write \( x = (P_N x, Q_N x) = (x_N, x_N^\perp) \).

We now fix \( N \in \mathbb{N} \). Then obviously

\[
\|x - y\|^2 = \|P_N(x - y)\|^2 + \|Q_N(x - y)\|^2,
\]

and we have

\[
\|Q_N(x - y)\|^2 \leq 2(Q_N(\hat{x} - \hat{y}), x - y) - \|Q_N(\hat{x} - \hat{y})\|^2 + 2\|Q_N(x - \hat{x})\|^2 + 2\|Q_N(y - \hat{y})\|^2
\]

with equality if \( x = \hat{x}, y = \hat{y} \). We define

\[
u_1(t, x) = \mu(t, x) - \frac{(x, Q_N(\hat{x} - \hat{y}))}{\epsilon} + \frac{\|Q_N(\hat{x} - \hat{y})\|^2}{2\epsilon}
\]

and

\[
v_1(s, y) = \frac{(y, Q_N(\hat{x} - \hat{y}))}{\epsilon} + \frac{\|Q_N(y - \hat{y})\|^2}{\epsilon} + \delta e^{K_{\mu}(T-t)}(1 + \|x\|^2)^m,
\]

We can consider \( u_1 \) and \( v_1 \) as functions defined on \((0, T) \times H \) by setting \( u_1(t, x) = -\infty \) and \( v_1(s, y) = +\infty \) if \( x, x \not\in V \). Such extended \( u_1 \) and \( v_1 \) are, respectively, weakly sequentially upper- and lower-semicontinuous on \((0, T) \times H \).
We now have that the function
\begin{equation}
\Phi(t, s, x, y) = u_1(t, x) - v_1(s, y) - \frac{\|P_N(x - y)\|^2}{2\epsilon} - \frac{(t - s)^2}{2\gamma}
\end{equation}
always satisfies \( \widetilde{\Phi} \leq \Phi \) and attains a strict global maximum at \( \hat{t}, \hat{s}, \hat{x}, \hat{y} \), where \( \widetilde{\Phi}(\hat{t}, \hat{s}, \hat{x}, \hat{y}) = \Phi(\hat{t}, \hat{s}, \hat{x}, \hat{y}) \).

Define, for \( x_N, y_N \in H_N \), the functions
\[ u_1(t, x_N) = \sup_{x_N^t \in H_N} u_1(t, x_N, x_N^t), \quad v_1(s, y_N) = \inf_{y_N^s \in H_N} v_1(s, y_N, y_N^s). \]
Since \( u_1 \) and \( -v_1 \) are weakly sequentially upper-semicontinuous on \( (0, T) \times H \), \( \hat{u}_1 \) and \( -\hat{v}_1 \) are upper semicontinuous on \( (0, T) \times H_N \) (see [3]). Moreover, it follows that
\begin{equation}
\hat{u}_1(\hat{t}, P_N \hat{x}) = u_1(\hat{t}, \hat{x}) \quad \hat{v}_1(\hat{s}, P_N \hat{y}) = v_1(\hat{s}, \hat{y}),
\end{equation}
and it is easy to see that the function
\[ \Phi(t, s, (x_N, x_N^t), (y_N, y_N^s)) \]
attains a strict global maximum over \( (0, T) \times (0, T) \times H_N \times H_N \) at \( (\hat{t}, \hat{s}, \hat{x}_N, \hat{y}_N) \).

By the finite-dimensional maximum principle (see [2]) for every \( n \in \mathbb{N} \) there exist points \( t^n, s^n \in (0, T) \) and \( x_n^t, y_n^s \in H_N \) such that
\begin{align}
& t^n \to \hat{t}, \quad s^n \to \hat{s}, \quad x_n^t \to \hat{x}_N, \quad y_n^s \to \hat{y}_N, \quad \text{as } n \to \infty, \\
& \hat{u}_1(t^n, x_n^t) \to \hat{u}_1(\hat{t}, \hat{x}_N), \quad \hat{v}_1(s^n, y_n^s) \to \hat{v}_1(\hat{s}, \hat{y}_N), \quad \text{as } n \to \infty,
\end{align}
and there exist functions \( \varphi_n, \psi_n \in C^{1,2}((0, T) \times H_N) \) with uniformly continuous derivatives such that \( \hat{u}_1 - \varphi_n \) and \( -\hat{v}_1 + \psi_n \) have strict, global maxima at \( (t^n, x_n^t) \) and \( (s^n, y_n^s) \), respectively, and
\begin{align}
& (\varphi_n)(t^n, x_n^t) \to \frac{\hat{t} - \hat{s}}{\gamma}, \quad D\varphi_n(t^n, x_n^t) \to \frac{1}{\epsilon} (\hat{x}_N - \hat{y}_N), \\
& (\psi_n)(s^n, y_n^s) \to \frac{\hat{t} - \hat{s}}{\gamma}, \quad D\psi_n(s^n, y_n^s) \to \frac{1}{\epsilon} (\hat{x}_N - \hat{y}_N), \\
& D^2\varphi_n(t^n, x_n^t) \to X_N, \quad D^2\psi_n(s^n, y_n^s) \to Y_N, \quad \text{where } X_N \leq Y_N.
\end{align}

We now consider the function
\begin{equation}
u_1(t, x) - v_1(s, y) - \varphi_n(t, P_N x) + \psi_n(s, P_N y).
\end{equation}
It attains its global maximum (which we can assume to be strict) at some point \( (\hat{t}^n, \hat{s}^n, \hat{x}^n, \hat{y}^n) \). Repeating now the arguments of [11, p. 409] (see also [3]) it is not difficult to show that
\begin{align}
& u_1(\hat{t}^n, \hat{x}^n) \to u_1(\hat{t}, \hat{x}), \quad v_1(\hat{s}^n, \hat{y}^n) \to u_1(\hat{s}, \hat{y}),
\end{align}
and

\[
\begin{align*}
\hat{t}^n &= t^n, \\
\hat{y}^n &= y^n, \\
\hat{x}^n &= x_N^n, \\
\hat{y}^n &= y_N^n,
\end{align*}
\]

(5.18)

\[(\hat{x}^n, \hat{y}^n) \to (\bar{x}, \bar{y}) \text{ in } H \times H \]
as \(n \to \infty\). Moreover,

\[
\|\hat{x}^n\|_1 \to \|\bar{x}\|_1, \quad \|\hat{y}^n\|_1 \to \|\bar{y}\|_1.
\]

But this then obviously implies that

\[
(5.19) \quad \hat{x}^n \to \bar{x}, \quad \hat{y}^n \to \bar{y}, \quad \text{in } V.
\]

We can now use that \(u_\mu\) and \(v_\mu\) are viscosity sub- and supersolutions. It follows from (5.16) and the definition of viscosity subsolution that

\[
\begin{align*}
- \delta K_{\mu} e^{K_{\mu}(T-t^n)} & \left(1 + \|\hat{x}^n\|_1^2\right)^m + (\varphi_n, (\hat{x}^n, \hat{y}^n)) \\
+ \frac{\delta}{2} e^{K_{\mu}(T-t^n)} & \left(2m \text{tr}(Q_1)(1 + \|\hat{x}^n\|_1^2)^{m-1}ight. \\
& \quad + 4m(m-1)(Q_1A^{1/2}\hat{x}^n, A^{1/2}\hat{x}^n)(1 + \|\hat{x}^n\|_1^2)^{m-2}\bigg) \\
- b(\hat{x}^n, \hat{x}^n, D\varphi_n(\hat{t}^n, \hat{x}^n) + \frac{Q_N(\hat{x} - \bar{x})}{\epsilon} + \frac{2Q_N(\hat{x} - \bar{x})}{\epsilon} \\
+ 2m \delta e^{K_{\mu}(T-t^n)} & \left(1 + \|\hat{x}^n\|_1^2\right)^{m-1}A\hat{x}^n \\
& \quad + F(\hat{t}^n, \hat{x}^n, D\varphi_n(\hat{t}^n, \hat{x}^n) + \frac{Q_N(\hat{x} - \bar{y})}{\epsilon} + \frac{2Q_N(\hat{x} - \bar{y})}{\epsilon} \\
& \quad + 2m \delta e^{K_{\mu}(T-t^n)} (1 + \|\hat{x}^n\|_1^2)^{m-1}A\hat{x}^n) \\
& \geq \frac{\mu}{T^2}.
\end{align*}
\]

(5.20)

We remark that we have used (2.3) to get \(b(\hat{x}^n, \hat{x}^n, A\hat{x}^n) = 0\). We now want to pass to the limit as \(n \to \infty\). Let \(C_{\mu}\) be a constant such that

\[
\omega(s) \leq \frac{\mu}{2T^2} + C_{\mu}s.
\]
It then follows from (5.2) that

\[
\left| F\left(\hat{t}^n, \hat{x}^n, D\varphi_n(\hat{t}^n, \hat{x}^n)\right) + \frac{Q_N(\hat{x} - \bar{y})}{\epsilon} + \frac{2Q_N(\hat{x}^n - \bar{x})}{\epsilon} + 2m\delta e^{K_\mu(T-\bar{t}^n)}(1 + \|\hat{x}^n\|_1^2)^{-1}\bar{A}\hat{x}^n \right|
\]

\[+ 2m\delta e^{K_\mu(T-\bar{t}^n)}(1 + \|\hat{x}^n\|_1^2)^{-1}\bar{A}\hat{x}^n \right| \leq \frac{\mu}{2T^2} + C_\mu(1 + \|\hat{x}^n\|_1)2m\delta e^{K_\mu(T-\bar{t}^n)}(1 + \|\hat{x}^n\|_1^2)^{-1}\|\bar{A}\hat{x}^n\|.
\]

Moreover,

\[
C_\mu(1 + \|\hat{x}^n\|_1)2m\delta e^{K_\mu(T-\bar{t}^n)}(1 + \|\hat{x}^n\|_1^2)^{-1}\|\bar{A}\hat{x}^n\|
\]

\[+ \frac{\delta}{2} e^{K_\mu(T-\bar{t}^n)}(2m \operatorname{tr}(Q_1)(1 + \|\hat{x}^n\|_1^2)^{-1}
\]

\[+ 4m(m - 1)(Q_1A^{1/2}\hat{x}^n, A^{1/2}\hat{x}^n)(1 + \|\hat{x}^n\|_1^2)^{m-2})
\]

\[\leq \frac{2m\delta}{v} C_\mu^2 e^{K_\mu(T-\bar{t}^n)}(1 + \|\hat{x}^n\|_1^2)^{m}
\]

\[+ v\delta e^{K_\mu(T-\bar{t}^n)}\|\bar{A}\hat{x}^n\|^2(1 + \|\hat{x}^n\|_1^2)^{m-1}
\]

\[+ \delta e^{K_\mu(T-\bar{t}^n)}m(2m - 1)\operatorname{tr}(Q_1)(1 + \|\hat{x}^n\|_1^2)^m
\]

\[= v\delta e^{K_\mu(T-\bar{t}^n)}\|\bar{A}\hat{x}^n\|^2(1 + \|\hat{x}^n\|_1^2)^{m-1}
\]

\[+ \delta e^{K_\mu(T-\bar{t}^n)}\left(\frac{2m}{v} C_\mu^2 + m(2m - 1)\operatorname{tr}(Q_1)\right)(1 + \|\hat{x}^n\|_1^2)^m.
\]

Therefore, choosing \(K_\mu = 1 + 2\left(\frac{2m}{v} C_\mu^2 + m(2m - 1)\operatorname{tr}(Q_1)\right)\), we obtain from (5.20) and (5.21) that

\[
- \frac{\delta}{2} K_\mu e^{K_\mu(T-\bar{t}^n)}(1 + \|\hat{x}^n\|_1^2)^m + (\varphi_n, (\hat{t}^n, \hat{x}^n))
\]

\[+ \frac{1}{2} \operatorname{tr} \left( QD^2\varphi_n(\hat{t}^n, \hat{x}^n) + 2QQ_N \right)
\]

\[- \left\langle v\bar{A}\hat{x}^n, D\varphi_n(\hat{t}^n, \hat{x}^n) + \frac{Q_N(\hat{x} - \bar{y})}{\epsilon} + \frac{2Q_N(\hat{x}^n - \bar{x})}{\epsilon} + m\delta e^{K_\mu(T-\bar{t}^n)}\bar{A}\hat{x}^n(1 + \|\hat{x}^n\|_1^2)^{-1} \right\rangle
\]

\[- b(\hat{x}^n, \hat{x}^n, D\varphi_n(\hat{t}^n, \hat{x}^n) + \frac{Q_N(\hat{x} - \bar{y})}{\epsilon} + \frac{2Q_N(\hat{x}^n - \bar{x})}{\epsilon})
\]

\[+ F\left(\hat{t}^n, \hat{x}^n, D\varphi_n(\hat{t}^n, \hat{x}^n) + \frac{Q_N(\hat{x} - \bar{y})}{\epsilon} + \frac{2Q_N(\hat{x}^n - \bar{x})}{\epsilon}\right)\]
(5.22) \[ \geq \frac{\mu}{2T^2}. \]

Now, using the boundedness of \( \|\hat{x}^a\|_1 \), (5.13) (which implies in particular that the \( \|D\varphi_\mu(\hat{\tau}^a, \hat{x}^a)\|_1 \) are bounded), (5.1), (5.2), and the continuity of \( b \) on \( V \times V \times V \), we obtain from (5.22) that the \( \|\hat{A}\hat{x}^a\| \) are bounded and therefore that \( \hat{x}^a \to \bar{x} \) in \( V_2 \). Thus, using (5.11), (5.13), (5.18), and (5.19), we can pass to the \( \lim \sup \) as \( n \to \infty \) in (5.22) to get
\[
-\frac{\delta}{2} K_\mu \epsilon^{K_\mu(T-\bar{\tau})} (1 + \|\bar{x}\|_1^2) + \frac{\bar{\tau} - \bar{\tau}}{\gamma} + \frac{1}{2} \text{tr}(QX_N + 2QQ_N)
\]
(5.23)
\[
- \left( \nu \hat{A} \hat{x}^a - \hat{y}^a \right) - b \left( \hat{x}, \hat{y}, \frac{\bar{x} - \bar{y}}{\epsilon} \right) + F \left( \hat{y}, \frac{\bar{x} - \bar{y}}{\epsilon} \right) \geq \frac{\mu}{2T^2}.
\]

Similarly, we obtain
\[
-\frac{\delta}{2} K_\mu \epsilon^{K_\mu(T-\bar{\tau})} (1 + \|\bar{y}\|_1^2) + \frac{\bar{\tau} - \bar{\tau}}{\gamma} + \frac{1}{2} \text{tr}(QY_N - 2QQ_N)
\]
(5.24)
\[
- \left( \nu \hat{A} \hat{y}^a - \hat{y}^a \right) - b \left( \hat{y}, \hat{y}, \frac{\bar{x} - \bar{y}}{\epsilon} \right) + F \left( \hat{y}, \frac{\bar{x} - \bar{y}}{\epsilon} \right) \leq -\frac{\mu}{2T^2}.
\]

Combining (5.23) and (5.24), using \( X_N \leq Y_N \), and then sending \( N \to \infty \) yields
\[
\frac{\delta}{2} \left( (1 + \|\bar{x}\|_1^2)^m + (1 + \|\bar{y}\|_1^2)^m \right) + \frac{\nu \|\bar{x} - \bar{y}\|_1^2}{\epsilon}
\]
\[
+ b \left( \frac{\bar{x} - \bar{y}}{\epsilon} \right) - b \left( \frac{\bar{y} - \bar{y}}{\epsilon} \right)
\]
\[
+ F \left( \frac{\bar{x} - \bar{y}}{\epsilon} \right) - F \left( \frac{\bar{y} - \bar{y}}{\epsilon} \right) \leq -\frac{\mu}{T^2}.
\]
(5.25)

To estimate the trilinear form terms, we use (2.2), (2.5), and then (5.7) to produce
\[
\left| b \left( \frac{\bar{x} - \bar{y}}{\epsilon} \right) - b \left( \frac{\bar{y} - \bar{y}}{\epsilon} \right) \right|
\]
\[
= \frac{1}{\epsilon} |b(\bar{x} - \bar{y}, \hat{x}, \bar{x} - \bar{y})| \leq \frac{1}{\epsilon} \|\bar{x}\|_1 \|\bar{x} - \bar{y}\|_1 \|\bar{x} - \bar{y}\|_1
\]
\[
\leq \frac{\delta}{2} \|\bar{x}\|_1^2 + C_\delta \frac{\|\bar{x} - \bar{y}\|_1^2 \|\bar{x} - \bar{y}\|_1^2}{\epsilon}
\]
\[
\leq \frac{\delta}{2} \left( 1 + \|\bar{x}\|_1^2 \right)^m + \sigma_1(\gamma, \epsilon; \delta, \mu) \frac{\|\bar{x} - \bar{y}\|_1^2}{\epsilon}
\]
(5.26)
for some local modulus \( \sigma_1 \).

Finally, we need to estimate the terms containing \( F \). We know that for \( \mu \) and \( \delta \) fixed, \( \|\bar{x}\|_1, \|\bar{y}\|_1 \leq R_3 \) for some \( R_3 > 0 \). Let \( D_{\mu, \delta} \) be a constant such that
\[
\omega_{R_3}(s) \leq \frac{\mu}{4T^2} + D_{\mu, \delta}s.
\]
Then
\[
|F\left(\bar{t}, \bar{x}, \frac{\bar{x} - \bar{y}}{\epsilon}\right) - F\left(\bar{s}, \bar{y}, \frac{\bar{x} - \bar{y}}{\epsilon}\right)|
\leq \omega_R(|\bar{t} - \bar{s}|) + \omega_R(\|\bar{x} - \bar{y}\|_1) + \omega\left(\|\bar{x} - \bar{y}\|_1 \frac{\|\bar{x} - \bar{y}\|}{\epsilon}\right)
\leq \sigma_2(\gamma; \epsilon, \delta, \mu) + \frac{3\mu}{4T^2} + D_{\mu, \delta}\|\bar{x} - \bar{y}\|_1 + C_\mu\|\bar{x} - \bar{y}\|_1 \frac{\|\bar{x} - \bar{y}\|}{\epsilon}
\leq \sigma_2(\gamma; \epsilon, \delta, \mu) + \frac{3\mu}{4T^2} + D_{\mu, \delta}\|\bar{x} - \bar{y}\|_1 + \frac{\nu\|\bar{x} - \bar{y}\|_1^2}{2\epsilon}
+ 2C_\mu^2 \frac{\|\bar{x} - \bar{y}\|^2}{\epsilon}
\leq \frac{3\mu}{4T^2} + \sigma_3(\gamma, \epsilon; \delta, \mu) + D_{\mu, \delta}\|\bar{x} - \bar{y}\|_1 + \frac{\nu\|\bar{x} - \bar{y}\|_1^2}{2\epsilon}.
\]

Therefore, plugging (5.26) and (5.27) into (5.25), we obtain
\[
(5.28) \left(\frac{\nu}{2} - \sigma_1(\gamma, \epsilon; \delta, \mu)\right) \frac{\|\bar{x} - \bar{y}\|_1^2}{\epsilon} - D_{\mu, \delta}\|\bar{x} - \bar{y}\|_1 \leq -\frac{\mu}{4T^2} + \sigma_4(\gamma, \epsilon, \delta; \mu).
\]

We now notice that if \( \gamma \) and \( \epsilon \) are small, then \( \frac{\nu}{2} - \sigma_1(\gamma, \epsilon; \delta, \mu) > \frac{\nu}{4} \) and that
\[
\liminf_{\epsilon \to 0, r > 0} \left(\frac{vr^2}{4\epsilon} - D_{\mu, \delta} r\right) = 0.
\]

Therefore it remains to send \( \gamma \to 0 \) and \( \epsilon \to 0 \) in this order in (5.28) to obtain a contradiction, which proves that we must have \( u \leq v \).

\[\square\]

### 6 Stochastic Optimal Control Problem and Existence of Viscosity Solutions

In this section we show that the value function of the stochastic optimal control problem is the unique viscosity solution of the associated HJB equation. We want to focus on the real difficulties of the problem that are caused by the stochastic Navier-Stokes equation. Therefore we will assume throughout the rest of the paper that the running cost function \( l \) is independent of \( t \). This is done to minimize technical difficulties that might obscure the main points of the proofs. The case of \( l \) depending on \( t \) can be handled by standard and rather straightforward methods; we leave the details to the interested reader.

There are no good continuous dependence estimates in the mean for solutions of the stochastic Navier-Stokes equation. To obtain such estimates, we need to be able to estimate exponential moments of solutions (3.4), and these seem to be bounded only for a short time (see [23, cor. XI.3.1]). Nevertheless, we are able to show the global continuity of the value function.

We make the following assumptions on the cost functions and the forcing function \( f \).
Hypothesis 6.1.

(i) The functions \( l : V \times U \rightarrow \mathbb{R} \) and \( g : V \rightarrow \mathbb{R} \) are continuous, and there exist \( k \geq 0 \) and, for every \( r > 0 \), a modulus \( \sigma_r \) such that

\[
(6.1) \quad |l(x, a)|, |g(x)| \leq C(1 + \|x\|_1^k) \quad \text{for all } x \in V, a \in U,
\]

\[
(6.2) \quad |l(x, a) - l(y, a)| \leq \sigma_r(||x - y||_1) \quad \text{if } ||x||_1, ||y||_1 \leq r, a \in U,
\]

\[
(6.3) \quad |g(x) - g(y)| \leq \sigma_r(||x - y||) \quad \text{if } ||x||_1, ||y||_1 \leq r.
\]

(ii) The function \( f : [0, T] \times U \rightarrow V \) is bounded, continuous, and \( f(\cdot, a) \) is uniformly continuous, uniformly for \( a \in U \).

Proposition 6.2 Let Hypothesis 6.1 be satisfied. Then for every \( r > 0 \) there exists a modulus \( \omega_r \) such that for every \( t \in [0, T] \) and \( \mathfrak{a}(\cdot) \in U_t \)

\[
(6.4) \quad |J(t, x; \mathfrak{a}(\cdot)) - J(t, y; \mathfrak{a}(\cdot))| \leq \omega_r(||x - y||) \quad \text{if } ||x||_1, ||y||_1 \leq r,
\]

and the value function

\[
\mathcal{V}(t, x) = \inf_{\mathfrak{a}(\cdot) \in U_t} J(t, x; \mathfrak{a}(\cdot))
\]

\[
= \inf_{\mathfrak{a}(\cdot) \in U_t} E_t \left\{ \int_t^T l(X(s; t, x, \mathfrak{a}(\cdot)), \mathfrak{a}(s))ds + g(X(T; t, x, \mathfrak{a}(\cdot))) \right\}
\]

satisfies

\[
(6.5) \quad |\mathcal{V}(t_1, x) - \mathcal{V}(t_2, y)| \leq \omega_r(|t_1 - t_2| + ||x - y||)
\]

for \( t_1, t_2 \in [0, T] \) and \( ||x||_1, ||y||_1 \leq r. \) Moreover,

\[
(6.6) \quad |\mathcal{V}(t, x)| \leq C(1 + ||x||_1^k).
\]

Proof: For every \( m > 0 \) let \( D_m \) be a constant such that \( \sigma_m(s) \leq \frac{1}{m} + D_ms \). Let \( t \in [0, T] \) and \( \mathfrak{a}(\cdot) \in U_t \). Denote \( X(s) = X(s; t, x, \mathfrak{a}(\cdot)), Y(s) = Y(s; t, y, \mathfrak{a}(\cdot)), A_m = \{ \omega \in \Omega_t : \max_{t \leq s \leq T} ||X(s)||_1 \leq m \}, B_m = \{ \omega \in \Omega_t : \max_{t \leq s \leq T} ||Y(s)||_1 \leq m \} \). Then, using (6.2), (3.4), and (3.3), we obtain (writing \( \chi_D \) for the indicator function of a set \( D \))

\[
\mathbb{E}_t \int_t^T |l(X(s), \mathfrak{a}(s)) - l(Y(s), \mathfrak{a}(s))|ds
\]

\[
\leq \frac{T}{m} + \mathbb{E}_t \int_t^T D_m ||X(s) - Y(s)||_1 \chi_{A_m \cap B_m} ds
\]

\[
+ \mathbb{E}_t \int_t^T C(1 + ||X(s)||_1^k + ||Y(s)||_1^k) \chi_{[\Omega_t \setminus (A_m \cap B_m)]} ds
\]
\[
\begin{align*}
&\leq \frac{T}{m} + D(m, T, v)\|x - y\|\mathbb{E}_t \int_t^T \exp \left( \frac{Cv}{2} \int_t^\nu \|X(\tau)\|^2 d\tau \right) \chi_{A_m} ds \\
&\quad + \int_t^T C(1 + \mathbb{E}_t\|X(s)\|^2)^{1/2} + (\mathbb{E}_t\|Y(s)\|^2)^{1/2}) \\
&\quad \times ((\mathbb{P}_r(\Omega_\nu \setminus A_m))^{1/2} + (\mathbb{P}_r(\Omega_\nu \setminus B_m))^{1/2}) ds \\
&\leq \frac{T}{m} + D(m, T, v)\|x - y\|e^{C(T-m)^2} \\
&\quad + C_1(1 + \|x\|_1^k + \|y\|_1^k)C(v, p, T, R, tr(Q_t)) \frac{1 + \|x\|_1 + \|y\|_1}{m}.
\end{align*}
\]

Applying the same process to estimate \(|g(X(T)) - g(Y(T))|\), we therefore obtain that for every \(r, m > 0\) there exist constants \(c_m\) and \(d\), such that for every \(t \in [0, T]\) and \(a(\cdot) \in \mathcal{U}_t\)

\[
|J(t, x; a(\cdot)) - J(t, y; a(\cdot))| \leq \frac{d}{m} + c_m \|x - y\| \quad \text{if} \quad \|x\|_1, \|y\|_1 \leq r.
\]

Estimate (6.4) now follows by taking the infimum over all \(m > 0\).

We also notice that (6.6) is a direct consequence of (6.1) and (3.2) and that (6.4) implies (6.5) if \(t_1 = t_2\).

Having (6.4) and the equicontinuity of the value function in \(x\) allows us to prove the dynamic programming principle. It states that

\[
\mathcal{V}(t, x) = \inf_{a(\cdot) \in \mathcal{U}_t} \mathbb{E}_t \left\{ \int_t^\eta l(X(s; t, x, a(\cdot)), a(s)) ds + \mathcal{V}(\eta, X(\eta; t, x, a(\cdot))) \right\}
\]

for every \(x \in \mathcal{V}\) and \(t \in [0, T]\), \(t \leq \eta \leq T\). The proof of (6.7) is given in the appendix.

We can now prove the continuity in the time variable \(t\). Let \(0 \leq t_1 < t_2 \leq T\) and \(x \in \mathcal{V}\). Using (6.7), (6.6), (3.2), (3.5), and (6.5) with fixed \(t\), we obtain for \(m \geq \|x\|_1\)

\[
|\mathcal{V}(t_1, x) - \mathcal{V}(t_2, x)|
\]

\[
\leq \sup_{a(\cdot) \in \mathcal{U}_{t_1}} \mathbb{E}_{t_1} \int_{t_1}^{t_2} (1 + \|X(s; t_1, x, a(\cdot))\|_1^k) ds
\]

\[
\quad + \sup_{a(\cdot) \in \mathcal{U}_{t_1}} \mathbb{E}_{t_1} |\mathcal{V}(t_2, X(t_2; t_1, x, a(\cdot))) - \mathcal{V}(t_2, x)|
\]

\[
\leq C(v, R, T, tr(Q_t), \|x\|_1) (t_2 - t_1)
\]

\[
\quad + \sup_{a(\cdot) \in \mathcal{U}_{t_1}} \left\{ \mathbb{E}_{t_1} \left[ C(1 + \|X(t_2; t_1, x, a(\cdot))\|_1^k + \|x\|_1^k) \\
\times \chi_{\|X(t_2; t_1, x, a(\cdot))\|_1 > m} \right] \right\}
\]

\[
\quad + \sup_{a(\cdot) \in \mathcal{U}_{t_1}} \mathbb{E}_{t_1} \sigma_m(\|X(t_2; t_1, x, a(\cdot)) - x\|)
\]
\begin{align}
\leq C(v, R, T, \text{tr}(Q_1), \|x\|_1)(t_2 - t_1) + C(v, R, T, \text{tr}(Q_1))(1 + \|x\|_1^k) \frac{1 + \|x\|_1}{m} \\
+ \sigma_m(C(v, R, \text{tr}(Q), \|x\|_1)(t_2 - t_1)^{1/2}).
\end{align} (6.8)

(We also used the fact that \(\sigma_m\) can be assumed to be concave.) The result now follows by taking the infimum over \(m\). \(\square\)

**Theorem 6.3** Let Hypothesis 6.1 be satisfied. Then the value function \(V\) is the unique viscosity solution of the HJB equation (1.4) satisfying (5.5) and (5.6).

**Proof:** First of all, we notice that if Hypothesis 6.1 is satisfied, then the Hamiltonian satisfies Hypothesis 5.1. Moreover, by Proposition 6.2 the value function \(V\) satisfies (5.6). Therefore the uniqueness of viscosity solutions is a direct consequence of Theorem 5.2. It remains to show that \(V\) is a viscosity solution. We will only show that the value function is a viscosity supersolution. The proof that \(V\) is a viscosity subsolution is easier and uses the same techniques. Let \(\psi(t, x) = \varphi(t, x) + \delta(t)(1 + \|x\|_1^k)^m\) be a test function, and let \(V + \psi\) have a global minimum at \((t_0, x_0) \in (0, T) \times \mathbb{V}\).

**Step 1.** We need to show that \(x_0 \in \mathbb{V}_2\). By the dynamic programming principle, for every \(\epsilon > 0\) there exists \(a_\epsilon(\cdot) \in \mathcal{U}_{t_0}\) such that writing \(X_\epsilon(s)\) for \(X(s; t_0, x_0, a_\epsilon(\cdot))\), we have

\[
V(t_0, x_0) + \epsilon^2 > \mathbb{E}_{t_0} \left\{ \int_{t_0}^{t_0+\epsilon} l(X_\epsilon(s), a_\epsilon(s))ds + V(t_0 + \epsilon, X_\epsilon(t_0 + \epsilon)) \right\}.
\]

Then, since for every \((t, x) \in (0, T) \times \mathbb{V}\)

\[
V(t, x) - V(t_0, x_0) \geq -\varphi(t, x) + \varphi(t_0, x_0) - \delta(t)(1 + \|x\|_1^k)^m + \delta(t_0)(1 + \|x_0\|_1^k)^m,
\]

we have

\[
\epsilon^2 - \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} l(X_\epsilon(s), a_\epsilon(s))ds
\]

\[
\geq \mathbb{E}_{t_0} V(t_0 + \epsilon, X_\epsilon(t_0 + \epsilon)) - V(t_0, x_0)
\]

\[
\geq \mathbb{E}_{t_0} (-\varphi(t_0 + \epsilon, X_\epsilon(t_0 + \epsilon)) + \varphi(t_0, x_0) - \delta(t_0 + \epsilon)(1 + \|X_\epsilon(t_0 + \epsilon)\|_1^k)^m + \delta(t_0)(1 + \|x_0\|_1^k)^m).
\]
Set \(\lambda = \inf_{t\in[0_0, t_0+\epsilon)} \delta(t)\) for some fixed \(\epsilon_0 > 0\) and take \(\epsilon < \epsilon_0\). Using Itô’s formula and (2.3) in the inequality above and then dividing both sides by \(\epsilon\), we obtain

\[
(6.9) \quad \epsilon - \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} l(X_\epsilon(s), a_\epsilon(s)) ds \\
\geq - \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \varphi_t(s, X_\epsilon(s)) ds \\
+ \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \{ vAX_\epsilon(s), D\varphi(s, X_\epsilon(s)) \} ds \\
+ \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \{ B(X_\epsilon(s), X_\epsilon(s)), D\varphi(s, X_\epsilon(s)) \} ds \\
+ \frac{1}{2\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \text{tr} (QD^2\varphi(s, X_\epsilon(s))) ds \\
- \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \{ f(s, a_\epsilon(s)), D\varphi(s, X_\epsilon(s)) \} ds \\
+ \frac{2m}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \delta(s) \{ vAX_\epsilon(s), AX_\epsilon(s) \} (1 + \|X_\epsilon(s)\|_1^2)^{m-1} ds \\
- \frac{2m}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \delta'(s) (1 + \|X_\epsilon(s)\|_1^2)^m ds \\
- \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \delta'(s) (1 + \|X_\epsilon(s)\|_1^2)^m ds \\
- \frac{m}{\epsilon} \text{tr}(Q_1) \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} (1 + \|X_\epsilon(s)\|_1^2)^{m-1} ds \\
- \frac{2m(m-1)}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} (1 + \|X_\epsilon(s)\|_1^2)^{m-2} (QAX_\epsilon(s), AX_\epsilon(s)) ds.
\]

By the definition of \(\lambda\), it then follows that

\[
(6.10) \quad \frac{2m\lambda v}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \|X_\epsilon(s)\|_2^2 (1 + \|X_\epsilon(s)\|_1^2)^{m-1} ds \\
\leq \epsilon - \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} l(X_\epsilon(s), a_\epsilon(s)) ds \\
+ \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \varphi_t(s, X_\epsilon(s)) ds \\
- \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \{ vAX_\epsilon(s), D\varphi(s, X_\epsilon(s)) \} ds \\
- \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \{ B(X_\epsilon(s), X_\epsilon(s)), D\varphi(s, X_\epsilon(s)) \} ds
\]
We now have

\[
\|I(X_\epsilon(s), a_\epsilon(s))\| \leq C + \|X_\epsilon(s)\|_1^2,
\]

(6.11)

\[
|\varphi_t(s, X_\epsilon(s))| \leq C + \|X_\epsilon(s)\|,
\]

(6.12)

\[
|\langle AX_\epsilon(s), D\varphi(s, X_\epsilon(s))\rangle| \leq \frac{\lambda}{2} \|X_\epsilon(s)\|_2^2 + C(1 + \|X_\epsilon(s)\|),
\]

(6.13)

\[
\left|\left|\langle B(X_\epsilon(s), X_\epsilon(s)), D\varphi(s, X_\epsilon(s))\rangle\right|\right| = |b(X_\epsilon(s), X_\epsilon(s), D\varphi(s, X_\epsilon(s)))|
\leq C \|X_\epsilon(s)\|_1 \|X_\epsilon(s)\|_2(1 + \|X_\epsilon(s)\|)
\leq \frac{\lambda v}{2} \|X_\epsilon(s)\|_2^2 + C(1 + \|X_\epsilon(s)\|_1^2),
\]

(6.14)

\[
|\text{tr}(QD^2\varphi(s, X_\epsilon(s)))|, \left|\left|\langle f(s, a_\epsilon(s)), D\varphi(s, X_\epsilon(s))\rangle\right|\right| \leq C(1 + \|X_\epsilon(s)\|),
\]

(6.15)

\[
|\langle f(s, a_\epsilon(s)), AX_\epsilon(s)\rangle\|_1(1 + \|X_\epsilon(s)\|_1^2)^{m-1} \leq C(1 + \|X_\epsilon(s)\|_1^2)^m,
\]

(6.16)

and

\[
(1 + \|X_\epsilon(s)\|_1^2)^{m-2}\left|\left|\langle QAX_\epsilon(s), AX_\epsilon(s)\rangle\right|\right| \leq C(1 + \|X_\epsilon(s)\|_1^2)^{m-1}.
\]

(6.17)

Employing inequalities (6.11)–(6.17) in (6.10) and then using (3.2) yields

\[
\frac{\lambda v}{\epsilon} \int_{t_0}^{t_0+\epsilon} \mathbb{E}_0 \|X_\epsilon(s)\|_2^2 ds \leq C
\]

for some constant C independent of \(\epsilon\). Therefore there exist sequences \(\epsilon_n \rightarrow 0\), \(t_n \in (t_0, t_0 + \epsilon_n)\), such that

\[
\mathbb{E}_0 \|X_{\epsilon_n}(t_n)\|_2^2 \leq C,
\]
and thus there exist subsequences, still denoted by $\epsilon_n$ and $t_n$, such that

$$X_{\epsilon_n}(t_n) \rightharpoonup \tilde{x} \text{ weakly in } L^2(\Omega_0, V_2)$$

for some $\tilde{x} \in L^2(\Omega_0, V_2)$. However, by (3.6), $X_{\epsilon_n}(t_n) \to x_0$ strongly in $L^2(\Omega_0, V)$. Therefore, by the uniqueness of the weak limit in $L^2(\Omega_0, V)$, it follows that $x_0 = \tilde{x} \in V_2$.

**Step 2.** We now prove the supersolution inequality. We need to “pass to the limit” as $\epsilon \to 0$ in (6.10), at least along a subsequence. This operation is rather standard for most of the terms, more precisely, for those that require only convergence in $H$ and $V$, because we can then use estimates (3.2) and (3.6).

For instance, for the cost term we argue that if $m \geq \|x_0\|_1$, then

$$\left| \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} [l(X_\epsilon(s), a_\epsilon(s))ds - l(x_0, a_\epsilon(s))]ds \right|$$

$$\leq \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \sigma_m(\|X_\epsilon(s) - x_0\|_1)ds$$

$$+ \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \sigma_m \left( \|X_\epsilon(s) - x_0\|_1 ds \right) + C \left( 1 + \|x_0\|_1^2 \right) \frac{1 + \|x_0\|_1}{m}$$

$$\leq \sigma_m \left( \frac{1}{\epsilon} \mathbb{E}_{t_0} \int_{t_0}^{t_0+\epsilon} \|X_\epsilon(s) - x_0\|_1 ds \right) + C \left( 1 + \|x_0\|_1^2 \right) \frac{1 + \|x_0\|_1}{m}$$

This implies that this term goes to 0 by letting $\epsilon \to 0$ and then $m \to \infty$.

The less standard terms are those containing $AX_\epsilon(s)$ and $B(X_\epsilon(s), X_\epsilon(s))$. To deal with them, we first notice that

$$\mathbb{E}_{t_0} \left\| \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} \sqrt{\delta(s)} AX_\epsilon(s) (1 + \|X_\epsilon(s)\|_1^2)^{m-1} ds \right\|^2$$

$$\leq \mathbb{E}_{t_0} \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} \delta(s) \|X_\epsilon(s)\|_2^2 (1 + \|X_\epsilon(s)\|_1^2)^{m-1} ds \leq C$$

by (6.10). Therefore, there exists a sequence $\epsilon_n \to 0$ and $y \in L^2(\Omega_0, H)$ such that

$$Y_n = \frac{1}{\epsilon_n} \int_{t_0}^{t_0+\epsilon_n} \sqrt{\delta(s)} AX_{\epsilon_n}(s) (1 + \|X_{\epsilon_n}(s)\|_1^2)^{m-1} ds \rightharpoonup y \text{ in } L^2(\Omega_0, H)$$
as $n \to \infty$. However, it is easy to see that
\[
\begin{align*}
A^{-1} Y_n &= \frac{1}{\epsilon_n} \int_{t_0}^{t_0 + \epsilon_n} \sqrt{\delta(s)} X_{\epsilon_n}(s) \left(1 + \|X_{\epsilon_n}(s)\|^2_1\right)^{n-1} ds \\
&\to \sqrt{\delta(t_0)} x_0 \left(1 + \|x_0\|_1^2\right)^{n-1} \text{ as } n \to \infty
\end{align*}
\]
strongly in $L^2(\Omega_{t_0}, H)$. Therefore it follows that
\[
Y = \sqrt{\delta(t_0)} A x_0 \left(1 + \|x_0\|_1^2\right)^{n-1}.
\]
Then, using the first part of (6.19), we get
\[
\liminf_{n \to \infty} \mathbb{E}_{t_0} \frac{1}{\epsilon_n} \int_{t_0}^{t_0 + \epsilon_n} \delta(s) \|X_{\epsilon_n}(s)\|^2_2 \left(1 + \|X_{\epsilon_n}(s)\|^2_1\right)^{n-1} ds \geq \delta(t_0) \|A x_0\|^2_2 \left(1 + \|x_0\|_1^2\right)^{n-1}. 
\]
The same argument also shows that we can assume that
\[
\frac{1}{\epsilon_n} \int_{t_0}^{t_0 + \epsilon_n} A X_{\epsilon_n}(s) ds \to A x_0 \text{ in } L^2(\Omega_{t_0}, H) \text{ as } n \to \infty.
\]
As regards other terms, denoting by $\omega_\varphi$ a modulus of continuity of $D \varphi$, we have
\[
\frac{1}{\epsilon_n} \mathbb{E}_{t_0} \int_{t_0}^{t_0 + \epsilon_n} \left[\langle A X_{\epsilon_n}(s), D \varphi(s, X_{\epsilon_n}(s))\rangle - \langle A x_0, D \varphi(t_0, x_0)\rangle\right] ds - \mathbb{E}_{t_0} \int_{t_0}^{t_0 + \epsilon_n} A X_{\epsilon_n}(s) ds - A x_0, D \varphi(t_0, x_0)\rangle \to 0 \text{ as } n \to \infty
\]
by (6.18), (3.5), and (6.21). Moreover, by (2.6),
\[
\frac{1}{\epsilon_n} \mathbb{E}_{t_0} \int_{t_0}^{t_0 + \epsilon_n} b(X_{\epsilon_n}(s), X_{\epsilon_n}(s), D \varphi(s, X_{\epsilon_n}(s))) ds - b(x_0, x_0, D \varphi(t_0, x_0))
\]
\[
\leq \frac{1}{\epsilon_n} \mathbb{E}_{t_0} \int_{t_0}^{t_0 + \epsilon_n} \|X_{\epsilon_n}(s)\|_1 \|X_{\epsilon_n}(s)\|_2 \omega_\varphi(\epsilon_n + \|X_{\epsilon_n}(s) - x_0\|) ds + \mathbb{E}_{t_0} \int_{t_0}^{t_0 + \epsilon_n} \|X_{\epsilon_n}(s) - x_0\|_1 \|X_{\epsilon_n}(s)\|_2 \|D \varphi(t_0, x_0)\| ds + \mathbb{E}_{t_0} \int_{t_0}^{t_0 + \epsilon_n} b(x_0, X_{\epsilon_n}(s) - x_0, D \varphi(t_0, x_0)) ds.
\]
\( \omega_n \) integral and then send and belong to the standard category since the map \( \Omega_1 \) have the canonical decomposition the operator \( A \) to 0 since for some modulus \( \omega \). We sketch its main points. The proof largely uses standard arguments. In particular, we follow the approach of [10] where the dynamic programming principle was proven for finite-dimensional stochastic differential games on the canonical sample space. The appendix is devoted to the proof of the dynamic programming principle (6.7). We sketch its main points. The proof largely uses standard arguments. In particular, we follow the approach of [10] where the dynamic programming principle was proven for finite-dimensional stochastic differential games on the canonical sample space.

First we make several observations. Denote \( \Omega_{t,\eta} = \{ \omega \in C([t, \eta]; H) : \omega(t) = 0 \} \) for \( 0 \leq t < \eta \leq T \), and denote by \( \mathbb{P}_{t,\eta} \) the Wiener measure on \( \Omega_{t,\eta} \). We then have the canonical decomposition \( \Omega_t = \Omega_{t,\eta} \times \Omega_\eta \), given by \( \omega = (\omega_1, \omega_2) \), where \( \omega_1 = \omega|_{[t,\eta]} \), \( \omega_2 = \omega - \omega(\eta)|_{[t,T]} \) (see [10]). Moreover, \( \mathbb{P}_t = \mathbb{P}_{t,\eta} \times \mathbb{P}_\eta \). We also

\[
(6.23) \quad \leq \frac{1}{\epsilon_n} \int_{t_0}^{t_0 + \epsilon_n} \left( \mathbb{E}_{t_0} \| \mathbf{X}_{\epsilon_n}(s) \|^2 \right)^{1/2} \left( \mathbb{E}_{t_0} \| \mathbf{X}_{\epsilon_n}(s) - \mathbf{x}_0 \|^2 \right)^{1/4} ds \times \left( \mathbb{E}_{t_0} (\omega_n + \| \mathbf{X}_{\epsilon_n}(s) - \mathbf{x}_0 \|)^4 \right)^{1/4} ds
\]

\[
+ C \frac{1}{\epsilon_n} \int_{t_0}^{t_0 + \epsilon_n} \left( \mathbb{E}_{t_0} \| \mathbf{X}_{\epsilon_n}(s) \|^2 \right)^{1/2} \left( \mathbb{E}_{t_0} \| \mathbf{X}_{\epsilon_n}(s) - \mathbf{x}_0 \|^2 \right)^{1/4} ds
\]

\[
+ \left| \mathbb{E}_{t_0} b \left( \mathbf{x}_0, \frac{1}{\epsilon_n} \int_{t_0}^{t_0 + \epsilon_n} \mathbf{X}_{\epsilon_n}(s) ds - \mathbf{x}_0, D\psi(t, \mathbf{X}_{\epsilon_n}(s)) \right) \right| \to 0
\]

as \( n \to \infty \) by (6.18), (3.2), (3.5), (3.6), and (6.21). In particular, the last term goes to 0 since

\[
\frac{1}{\epsilon_n} \int_{t_0}^{t_0 + \epsilon_n} \mathbf{X}_{\epsilon_n}(s) ds \to \mathbf{x}_0 \quad \text{in } L^2(\Omega_{t_0}, V_2) \text{ as } n \to \infty,
\]

and

\[
Y \to \mathbb{E}_{t_0} b(\mathbf{x}_0, Y, D\psi(t_0, \mathbf{x}_0))
\]

is a bounded linear functional on \( L^2(\Omega_{t_0}, V_2) \).

Finally, we notice that the terms containing \( (f(s, \mathbf{x}_\epsilon(s)), AX_\epsilon(s)) \) and \( (QAX_\epsilon(s), AX_\epsilon(s)) \) belong to the standard category since the map \( A^{1/2}f \) has values bounded in \( H \) and the operator \( A^{1/2}QA^{1/2} \) is bounded in \( H \).

Therefore, using (6.20), (6.22), (6.23), and other similar estimates in (6.9), we obtain for small \( \epsilon_n \) that

\[
-\psi(t, \mathbf{x}_0) - \frac{1}{2} \text{tr}(Q D^2 \psi(t, \mathbf{x}_0)) + \langle Ax_0 + B(\mathbf{x}_0, \mathbf{x}_0), D\psi(t, \mathbf{x}_0) \rangle
\]

\[
+ \frac{1}{\epsilon_n} \mathbb{E}_{t_0} \int_{t_0}^{t_0 + \epsilon_n} \left[ (f(t_0, \mathbf{a}_\epsilon(s)), -D\psi(t_0, \mathbf{x}_0)) + l(\mathbf{x}_0, \mathbf{a}_\epsilon(s)) \right] ds \leq \omega_1(\epsilon_n)
\]

for some modulus \( \omega_1 \). It now remains to take the infimum over \( \mathbf{a} \in U \) inside the integral and then send \( n \to \infty \).

\[
\Box
\]

**Appendix: Proof of Dynamic Programming Principle**

The appendix is devoted to the proof of the dynamic programming principle (6.7). We sketch its main points. The proof largely uses standard arguments. In particular, we follow the approach of [10] where the dynamic programming principle was proven for finite-dimensional stochastic differential games on the canonical sample space.

First we make several observations. Denote \( \Omega_{t,\eta} = \{ \omega \in C([t, \eta]; H) : \omega(t) = 0 \} \) for \( 0 \leq t < \eta \leq T \), and denote by \( \mathbb{P}_{t,\eta} \) the Wiener measure on \( \Omega_{t,\eta} \). We then have the canonical decomposition \( \Omega_t = \Omega_{t,\eta} \times \Omega_\eta \), given by \( \omega = (\omega_1, \omega_2) \), where \( \omega_1 = \omega|_{[t,\eta]} \), \( \omega_2 = \omega - \omega(\eta)|_{[t,T]} \) (see [10]). Moreover, \( \mathbb{P}_t = \mathbb{P}_{t,\eta} \times \mathbb{P}_\eta \). We also
notice that it follows from the definition of admissible controls that for 0 ≤ t < \eta < T, if \textbf{a}(\cdot) \in \mathcal{U}_t, then for a.e. \omega_1 \in \Omega_{t,\eta}, the map \textbf{a}_{\omega_1} : [\eta, T] \times \Omega_\eta \to \textbf{U}
given by \textbf{a}_{\omega_1}(\omega_2) = \textbf{a}(\omega_2, \omega_2) \text{ belongs to } \mathcal{U}_\eta. \text{ Second, we observe that if } \textbf{X}(\cdot) = \textbf{X}(\cdot; t, \textbf{x}, \textbf{a}(\cdot)), \textbf{x} \in \textbf{V}, \text{ and } t < \eta < T, \text{ then, denoting by } \textbf{W}_t \text{ and } \textbf{W}_\eta \text{ the Wiener processes on } \Omega_t \text{ and } \Omega_\eta, \text{ respectively,}

\[
\begin{align*}
\textbf{X}(s) &= \\
&= \textbf{X}(\eta) + \int_\eta^s (-\textbf{A}\textbf{X}(r) - \textbf{B}(\textbf{X}(r), \textbf{X}(r)) + \textbf{f}(r, \textbf{a}(r))) dr + \textbf{W}_t(s) - \textbf{W}_t(\eta).
\end{align*}
\]

Since for \(P_{t,\eta} \text{ a.e. } \omega_1\)

\[
\textbf{W}_t(s)(\omega_1, \omega_2) - \textbf{W}_t(\eta)(\omega_1, \omega_2) = \omega_2(s) = \textbf{W}_\eta(s)(\omega_2),
\]

we obtain that for \(P_{t,\eta} \text{ a.e. } \omega_1, \textbf{X}(s) \text{ is equal to the solution of (1.3) on } [\eta, T] \text{ with initial condition } \textbf{X}(\eta)(\omega_1) \text{ and control } \textbf{a}_{\omega_1}(\cdot). \text{ (We slightly abuse notation here, but since } \textbf{W}_t(\eta)(\omega_1, \omega_2) = \omega_1(\eta), \text{ it is clear that } \textbf{X}(\eta) \text{ is independent of } \omega_2.) \text{ Noticing also that}

\[
\begin{align*}
\mathbb{E}_t[g(\textbf{X}(T)(\omega_1, \omega_2)) \mid \mathcal{F}_{t,\eta}] &= \mathbb{E}_\eta g(\textbf{X}(T)(\omega_1, \omega_2)) \quad \text{for } P_{t,\eta} \text{ a.e. } \omega_1
\end{align*}
\]

we obtain that

\[
\begin{align*}
\mathbb{E}_t[g(\textbf{X}(T; t, \textbf{x}, \textbf{a}(\cdot))) \mid \mathcal{F}_{t,\eta}] &= \\
&= \mathbb{E}_\eta g(\textbf{X}(T; \eta, \textbf{X}(\eta)(\omega_1), \textbf{a}_{\omega_1}(\cdot))) \quad \text{for } P_{t,\eta} \text{ a.e. } \omega_1.
\end{align*}
\]

The same reasoning applied to \(l\) gives that for \(\eta \leq s \leq T\)

\[
\begin{align*}
\mathbb{E}_t[l(\textbf{X}(s; t, \textbf{x}, \textbf{a}(\cdot)), \textbf{a}(s)) \mid \mathcal{F}_{t,\eta}] &= \mathbb{E}_\eta l(\textbf{X}(s; \eta, \textbf{X}(\eta)(\omega_1), \textbf{a}_{\omega_1}(\cdot)), \textbf{a}_{\omega_1}(s))
\end{align*}
\]

for \(P_{t,\eta} \text{ a.e. } \omega_1. \text{ Therefore we obtain that for } P_{t,\eta} \text{ a.e. } \omega_1\)

\[
\begin{align*}
J(\eta, \textbf{X}(\eta)(\omega_1); \textbf{a}_{\omega_1}(\cdot)) &= \\
&= \mathbb{E}_t \left[ \int_\eta^T l(\textbf{X}(s; t, \textbf{x}, \textbf{a}(\cdot)), \textbf{a}(s)) ds + g(\textbf{X}(T; t, \textbf{x}, \textbf{a}(\cdot))) \bigg| \mathcal{F}_{t,\eta} \right]
\end{align*}
\]

for every \(t < \eta \leq T.\)

**PROOF OF DYNAMIC PROGRAMMING PRINCIPLE:** Throughout the proof we will write \(\omega = (\omega_1, \omega_2) \in \Omega_{t,\eta} \times \Omega_\eta. \text{ Let } \epsilon > 0, \text{ and let } \textbf{a}^\epsilon(\cdot) \text{ be an } \epsilon \text{-optimal control for } (t, \textbf{x}). \text{ Then, by (A.1),}

\[
\begin{align*}
\mathcal{V}(t, \textbf{x}) &+ \infty \\
\geq \mathbb{E}_t \left\{ \int_\eta^T l(\textbf{X}(s; t, \textbf{x}, \textbf{a}^\epsilon(\cdot)), \textbf{a}^\epsilon(s)) ds \right\} \\
&+ \mathbb{E}_t \left\{ \int_\eta^T l(\textbf{X}(s; t, \textbf{x}, \textbf{a}^\epsilon(\cdot)), \textbf{a}^\epsilon(s)) ds + g(\textbf{X}(T; t, \textbf{x}, \textbf{a}^\epsilon(\cdot))) \right\}
\end{align*}
\]
\[ \begin{align*}
&= \mathbb{E}_t \left\{ \int_t^\eta l(X(s; t, x, a^\ell(\cdot), a^\ell(s))ds \right\} \\
&\quad + \mathbb{E}_t \left[ \mathbb{E}_t \left[ \int_t^T l(X(s; t, x, a^\ell(\cdot), a^\ell(s))ds + g(X(T; t, x, a^\ell(\cdot))) \right| \mathcal{F}_{t, \eta} \right] \right] \\
&\geq \mathbb{E}_t \left\{ \int_t^\eta l(X(s; t, x, a^\ell(\cdot), a^\ell(s))ds \right\} \\
&\quad + \mathbb{E}_t J(\eta, X(\eta; t, x, a^\ell(\cdot))(\omega_1); a^\ell_{\omega_1}(\cdot)) \\
&\geq \mathbb{E}_t \left\{ \int_t^\eta l(X(s; t, x, a^\ell(\cdot), a^\ell(s))ds + \mathcal{V}(\eta, X(\eta; t, x, a^\ell(\cdot))) \right\}.
\end{align*} \]

Taking the infimum over all \( a(\cdot) \in \mathcal{U}_t \) and then letting \( \epsilon \to 0 \) now gives
\[ \mathcal{V}(t, x) \geq \inf_{a(\cdot) \in \mathcal{U}_t} \mathbb{E}_t \left\{ \int_t^\eta l(X(s; t, x, a(\cdot), a(s))ds + \mathcal{V}(\eta, X(\eta; t, x, a(\cdot))) \right\}. \]

It remains to prove the opposite inequality. To this end let \( \epsilon > 0 \). By (6.5) and (6.4), there exists a countable partition of \( V \) into disjoint Borel sets \( \{D_j\}_{j \in \mathbb{N}} \) such that if \( a(\cdot) \in \mathcal{U}_0 \) and \( x, y \in D_j \), then
\[ \text{(A.2)} \quad |J(\eta, x; a(\cdot)) - J(\eta, y; a(\cdot))| + |\mathcal{V}(\eta, x) - \mathcal{V}(\eta, y)| \leq \epsilon. \]

Choose for every \( j \in \mathbb{N} \) a point \( x_j \in D_j \) and a control \( a_j(\cdot) \in \mathcal{U}_\eta \) such that
\[ \text{(A.3)} \quad J(\eta, x_j; a_j(\cdot)) \leq \mathcal{V}(\eta, x_j) + \epsilon. \]

For a given \( a(\cdot) \in \mathcal{U}_t \) we can now define a control \( \tilde{a}(\cdot) \) by
\[ \tilde{a}(s)(\omega) = \begin{cases} a(s)(\omega) & \text{if } t \leq s < \eta \\ \sum_{j \in \mathbb{N}} a_j(s)(\omega_2) \chi_{\{X(\eta; t, x, a(\cdot))\in D_j\}} & \text{if } s \geq \eta. \end{cases} \]

It is easy to see that \( \tilde{a}(\cdot) \in \mathcal{U}_t \). Therefore, by (A.1),
\[ \mathcal{V}(\eta, x) \]
\[ \leq \mathbb{E}_t \left\{ \int_t^\eta l(X(s; t, x, \tilde{a}(\cdot)), \tilde{a}(s))ds \right\} \\
+ \mathbb{E}_t \left[ \mathbb{E}_t \left[ \int_t^T l(X(s; t, x, \tilde{a}(\cdot)), \tilde{a}(s))ds + g(X(T; t, x, \tilde{a}(\cdot))) \right| \mathcal{F}_{t, \eta} \right] \right] \\
\leq \mathbb{E}_t \left\{ \int_t^\eta l(X(s; t, x, a(\cdot)), a(s))ds \right\} \\
+ \sum_{j \in \mathbb{N}} \mathbb{E}_t J(\eta, X(\eta; t, x, a(\cdot)); a_j(\cdot)) \chi_{\{X(\eta; t, x, a(\cdot))\in D_j\}}. \]

However, if \( X(\eta; t, x, a(\cdot)) \in D_j \), then by (A.2) and (A.3)
\[ J(\eta, X(\eta; t, x, a(\cdot)); a_j(\cdot)) \leq J(\eta, x_j; a_j(\cdot)) + \epsilon \]
\[ \leq \mathcal{V}(\eta, x_j) + 2\epsilon \leq \mathcal{V}(\eta, X(\eta; t, x, a(\cdot))) + 3\epsilon. \]
Therefore it follows that
\[ V(\eta, x) \leq \mathbb{E}\left\{ \int_t^\tau l(X(s; t, x, a(\cdot)), a(s))\,ds + V(\eta, X(\eta; t, x, a(\cdot))) \right\} + 3\epsilon. \]

Since \( a(\cdot) \) was arbitrary, we can now take the infimum over all \( a(\cdot) \in \mathcal{U}_t \) in the above and then let \( \epsilon \to 0 \) to obtain the required inequality. \( \square \)

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