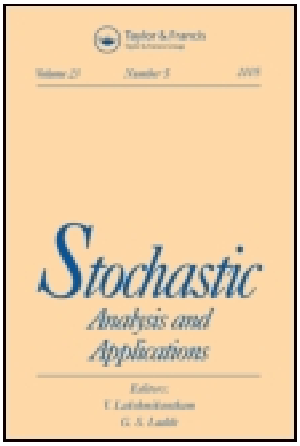

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Nonlinear Filtering of Stochastic Navier-Stokes Equation with Itô-Lévy Noise

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Nonlinear Filtering of Stochastic Navier-Stokes Equation with Itô-Lévy Noise

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In this article, we study the existence and uniqueness of the strong pathwise solution of stochastic Navier-Stokes equation with Itô-Lévy noise. Nonlinear filtering problem is formulated for the recursive estimation of conditional expectation of the flow field given back measurements of sensor output data. The corresponding Fujisaki-Kallianpur-Kunita and Zakai equations describing the time evolution of the nonlinear filter are derived. Existence and uniqueness of measure-valued solutions are proven for these filtering equations.

Keywords Fujisaki-Kallianpur-Kunita equation; Itô-Lévy noise; Nonlinear filtering; Stochastic Navier-Stokes equation; Zakai equation.

Mathematics Subject Classification 35R60; 93E11; 35Q30.

1. Introduction

Nonlinear filtering for fluid dynamic system has a wide range of applications in many fields in engineering sciences such as turbulence diagnostics, weather prediction, and oceanography. This subject was initiated in [38–42] and in [15] a reacting and diffusing system was studied. In this work, we derive the Fujisaki-Kallianpur-Kunita (FKK) equation and the Zakai equation for the nonlinear filtering of stochastic Navier-Stokes equation with Itô-Lévy noise, and prove existence and uniqueness of the measure-valued solution in certain class of measures.

We consider stochastic Navier-stokes equation with multiplicative white noise and jump noise. Existence of strong pathwise solutions are proven by Galerkin approximation and weak limit with a modification of Minty-Browder technique as in [9, 26, 37], and [43]. We show the existence and uniqueness of strong pathwise solutions for the case of σ-finite Lévy measure by constructing an approximate

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sequence of stochastic Navier-Stokes equations with Itô-Lévy measure of finite Lévy measure type.

In [38], existence and uniqueness of the measure-valued solution of FKK and Zakai equation for stochastic Navier-stokes equation with white noise was established for the case of unbounded observation. In this work, we generalize this result by adding Lévy noise to the signal process.

The structure of this article is as follows. In the next section, we describe some mathematical properties of Navier-Stokes operators (see [22, 23]), related function spaces and the coefficients of Itô-Lévy noise. We note a local monotonicity property of the Navier-Stokes operators and coefficient functions of Itô-Lévy noise. Then we prove a priori estimates for the stochastic Navier-Stokes equation. In later part of this section, we show the existence and uniqueness of strong pathwise solutions of stochastic Navier-Stokes equation for the cases of finite and σ-finite Lévy measure. In Section 3, we derive the FKK and Zakai equations and prove existence and uniqueness of the measure valued solutions.

2. Stochastic Navier-Stokes Equation with Itô-Lévy Noise

2.1. *Preliminaries*

Let $\mathrm{G} \subset \mathbb{R}^2$ be an arbitrary, possibly unbounded open domain with a smooth boundary $\partial\mathrm{G}$ if the domain has a boundary. Let us denoted by $\mathbf{u}$ and p as the velocity and pressure fields respectively. The stochastic Navier-Stokes equation with Itô-Lévy noise is formulated as follows.

$$\begin{aligned} &\mathrm{d}\mathbf{u}(y,t) + [-\nu\triangle\mathbf{u}(y,t) + (\mathbf{u}(y,t)\cdot\nabla)\mathbf{u}(y,t) + \nabla p(y,t)]\mathrm{d}t \\ &\quad = \mathbf{f}(y,t)\mathrm{d}t + \psi(t,\mathbf{u}(y,t))\mathrm{dW}(y,t) \\ &\qquad + \int_{\mathbb{H}} \phi(\mathbf{u}(y,t^-),x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) \text{ in } \mathrm{G}\times(0,T), \end{aligned} \tag{2.1}$$

with the conditions

$$\begin{cases} \nabla\cdot\mathbf{u}(y,t) = 0 & \text{in } \mathrm{G}\times(0,T) \\ \mathbf{u}(y,t) = 0 & \text{in } \partial\mathrm{G}\times(0,T) \\ \mathbf{u}(y,0) = \mathbf{u}_0(y) & \text{in } \mathrm{G}\times\{0\}, \end{cases}$$
$$\mathbf{u}(y,t)\to 0 \text{ as } |y|\to\infty \text{ if G is unbounded.} \tag{2.2}$$

In the system (2.1)–(2.2), $\mathbf{f}:\mathrm{G}\times(0,T)\to\mathbb{R}^2$ is a possibly random forcing term, $\nu>0$ is the coefficient of kinematic viscosity. $\mathrm{W}(.,t)$ is a Hilbert-space valued Wiener process [8] in time with the trace-class covariance and $\widetilde{N}(\mathrm{d}t,\mathrm{d}x)$ is the compensated Poisson measure [2]. We assume that $\mathrm{W}(.,.)$ and $N(.,.)$ are independent. $\psi(.,.)$ and $\phi(.,.)$ represent the multiplicative diffusion coefficient function and the jump noise coefficient function respectively. Assumptions regarding $\psi(.,.)$ and $\phi(.,.)$ are given in this section.

Let us express the Stochastic Navier-Stokes equation (2.1) in the abstract form using the following function spaces (for definition of these spaces see [44–46] for bounded domains and [12, 21, 24, 36, 44, 47] for arbitrary domains):

Let $\mathscr{V} = \{\mathbf{u}\in\mathbb{C}_0^\infty(\mathrm{G}) \mid \nabla\cdot\mathbf{u}=0\}$. The spaces $\mathbb{H}$ and $\mathbb{V}$ are defined by the completion of $\mathscr{V}$ with $\mathbb{L}^2(\mathrm{G})$ and $\mathbb{H}^1(\mathrm{G})$ norms, respectively. The space $\overline{\mathbb{V}}$ is defined by the completion of $\mathscr{V}$ in the seminorm $|\nabla \mathbf{u}|_{\mathbb{L}^2}$. In general $\mathbb{V}$ and $\overline{\mathbb{V}}$ are different, but they would coincide when G is a Poincaré domain [44]. In the case of bounded domains we have

$$\mathbb{H} := \{\mathbf{u} \in \mathbb{L}^2(\mathrm{G}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial \mathrm{G}} = 0\}, \tag{2.3}$$

$$\mathbb{V} := \left\{\mathbf{u} \in \mathbb{W}_0^{1,2}(\mathrm{G}) : \nabla \cdot \mathbf{u} = 0\right\}, \tag{2.4}$$

where $\mathbb{W}_0^{1,2}(\mathrm{G}) = \{\mathbf{u} \in \mathbb{L}^2(\mathrm{G}) : \nabla \mathbf{u} \in \mathbb{L}^2(\mathrm{G}), \mathbf{u}|_{\partial \mathrm{G}} = 0\}$ and $\mathbf{n}$ is the outward normal vector. By taking $\mathbb{V}'$ as the dual of $\mathbb{V}$, we will have the dense continuous embedding $\mathbb{V} \subset_{\rightarrow} \mathbb{H} = \mathbb{H}' \subset_{\rightarrow} \mathbb{V}'$.

Let us define Stokes operator $\mathbf{A}$ and nonlinear operator $\mathbf{B}$ as follows.

$$\mathbf{A} : \mathbb{H}^2(\mathrm{G}) \cap \mathbb{V} \to \mathbb{H}, \quad \text{with } \mathbf{A}\mathbf{u} = -\nu P_{\mathrm{H}} \Delta \mathbf{u}, \tag{2.5}$$

$$\mathbf{B} : \mathrm{D}(\mathbf{B}) \subset \mathbb{H} \times \mathbb{V} \to \mathbb{H}, \quad \text{with } \mathbf{B}(\mathbf{u}, \mathbf{v}) = P_{\mathrm{H}}(\mathbf{u} \cdot \nabla \mathbf{v}). \tag{2.6}$$

Here $P_{\mathrm{H}} : \mathrm{L}^2(\mathrm{G}) \to \mathrm{H}$ is the Helmholtz-Hodge projection.

The norms in the Hilbert spaces $\mathbb{H}$, and $\mathbb{V}$ are denoted by $|.|$, and $\|.\|$ respectively. The inner product in the Hilbert space $\mathbb{H}$ and the induced duality associate with $\mathbb{V}$ and $\mathbb{V}'$ are denoted by $(.,.)$ and $\langle .,. \rangle$ respectively. For $\mathbf{u} = (\mathrm{u}_i)$, $\mathbf{v} = (\mathrm{v}_i)$ and $\mathbf{w} = (\mathrm{w}_i)$, we have

$$\langle \mathbf{A}\mathbf{u}, \mathbf{w} \rangle = \nu \textstyle\sum_{i,j} \int_{\mathrm{G}} \partial_i \mathrm{u}_j \partial_i \mathrm{w}_j dx, \tag{2.7}$$

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = \textstyle\sum_{i,j} \int_{\mathrm{G}} \mathrm{u}_i (\partial_i \mathrm{v}_j) \mathrm{w}_j dx. \tag{2.8}$$

By applying integration by parts to (2.8), we will get

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = -\langle \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v} \rangle, \tag{2.9}$$

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle = 0, \tag{2.10}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$.

Lemma 2.1. *The trilinear form* $\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = \sum_{i,j} \int_{\mathrm{G}} \mathrm{u}_i (\partial_i \mathrm{v}_j) \mathrm{w}_j dx$ *is continuous on* $\mathbb{H}^{m_1}(\mathrm{G}) \times \mathbb{H}^{m_2+1}(\mathrm{G}) \times \mathbb{H}^{m_3}(\mathrm{G})$ *where* $m_i \geq 0$, *and*

$$m_1 + m_2 + m_3 \geq \frac{n}{2} \quad \text{if } m_i \neq \frac{n}{2}, \quad \text{for all } i.$$

$$m_1 + m_2 + m_3 > \frac{n}{2} \quad \text{if } m_i = \frac{n}{2}, \quad \text{for some } i.$$

Furthermore, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ *and* $n = 2$,

$$|\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle| \leq K_B |\mathbf{u}|^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{1}{2}} \|\mathbf{v}\| |\mathbf{w}|^{\frac{1}{2}} \|\mathbf{w}\|^{\frac{1}{2}}. \tag{2.11}$$

Proof. See Lemma 2.1 in [46]. □

Let Q be a symmetric, positive trace class operator on $\mathbb{H}$. Then there exist a sequence of eigenvalues $\{\gamma_k\}$ with the corresponding sequence of eigenvectors $\{e_k\}$ such that $\mathrm{Tr}(\mathrm{Q}) = \sum_{k=1}^{\infty} \gamma_k < \infty$ and $\mathrm{Q} e_k = \gamma_k e_k$ for all $k \in \mathbb{N}$ (see [8]).

Definition 2.1. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathrm{P})$ be a complete, filtered probability space. A stochastic process $\{\mathrm{W}(t) : t > 0\}$ is said to be a $\mathbb{H}$-valued $\mathscr{F}_t$-adapted Wiener process with covariance operator Q if the following two conditions are satisfied:

1. For each non-zero $\mathrm{h} \in \mathbb{H}$, $|\mathrm{Q}^{1/2}\mathrm{h}|^{-1}(\mathrm{W}(t), \mathrm{h})$ is a standard one-dimensional Wiener process.
2. For any $\mathrm{h} \in \mathbb{H}$, $(\mathrm{W}(t), \mathrm{h})$ is a martingale adapted to $\mathscr{F}_t$.

Let $\mathrm{L_Q}$ denote the space of linear bounded operators S such that $\mathrm{SQ}^{1/2}$ is Hilbert-Schmidt from $\mathbb{H}$ to $\mathbb{H}$. That is for any orthonormal basis $\{e_k\}$ in $\mathbb{H}$, $\sum_{k=1}^{\infty} |\mathrm{SQ}^{1/2}e_k|^2 < \infty$. The norm of $\mathrm{L_Q}$ obtained as follows.

$$
\begin{aligned}
|\mathrm{S}|_{\mathrm{L_Q}}^2 &= \sum_{k=1}^{\infty} |\mathrm{SQ}^{1/2}e_k|^2 = \sum_{k=1}^{\infty} \left(\mathrm{SQ}^{1/2}e_k, \mathrm{SQ}^{1/2}e_k\right) \\
&= \sum_{k=1}^{\infty} \left(\mathrm{Q}^{1/2}\mathrm{S}^*\mathrm{SQ}^{1/2}e_k, e_k\right) = \mathrm{Tr}((\mathrm{SQ}^{1/2})^*\mathrm{SQ}^{1/2}) \\
&= \mathrm{Tr}(\mathrm{SQ}^{1/2}(\mathrm{SQ}^{1/2})^*) = \mathrm{Tr}(\mathrm{SQS}^*).
\end{aligned} \tag{2.12}
$$

In the above, we used the fact that, if A is a Hilbert-Schmidt operator then $\mathrm{Tr}(\mathrm{A}^*\mathrm{A}) = \mathrm{Tr}(\mathrm{AA}^*)$.

We shall impose the following hypotheses on multiplicative noise coefficient $\psi : [0, T] \times \mathbb{H} \to \mathrm{L}(\mathbb{H}; \mathbb{H})$.

1. There exists a positive constant $\widetilde{K}_1$ such that

$$|\psi(t, \mathbf{u})|_{\mathrm{L}(\mathbb{H};\mathbb{H})} \le \widetilde{K}_1(1 + |\mathbf{u}|), \quad \text{for all } t \in [0, T],\ \mathbf{u} \in \mathbb{H}.$$

2. There exists a positive constant $\widetilde{M}_1$ such that

$$|\psi(t, \mathbf{u}) - \psi(t, \mathbf{v})|_{\mathrm{L}(\mathbb{H};\mathbb{H})} \le \tilde{M}_1|\mathbf{u} - \mathbf{v}|, \quad \text{for all } t \in [0, T],\ \mathbf{u}, \mathbf{v} \in \mathbb{H}.$$

Now we can obtain the main assumptions (A1 and A2) regarding $\psi : [0, T] \times \mathbb{H} \to \mathrm{L}(\mathbb{H}; \mathbb{H})$ from the above two hypothesis. Consider

$$
\begin{aligned}
|\psi(t, \mathbf{u})|_{\mathrm{L_Q}}^2 &= \sum_{k=1}^{\infty} (\mathrm{Q}^{1/2}\psi^*(t, \mathbf{u})\psi(t, \mathbf{u})\mathrm{Q}^{1/2}e_k, e_k) \\
&= \sum_{k=1}^{\infty} \gamma_k|\psi(t, \mathbf{u})e_k|^2 \le \sum_{k=1}^{\infty} \gamma_k|\psi(t, \mathbf{u})|_{\mathrm{L}(\mathbb{H};\mathbb{H})}^2|e_k|^2 \le \acute{M}|\psi(t, \mathbf{u})|_{\mathrm{L}(\mathbb{H};\mathbb{H})}^2 \\
&\le \mathrm{K}_1(1 + |\mathbf{u}|^2),
\end{aligned} \tag{2.13}
$$

where $\gamma_1, \gamma_2, \ldots$ are the eigenvalues of the trace class operator Q. Similarly we can obtain assumption A2 from Hypothesis 2. Now we have main assumptions regarding $\psi : [0, T] \times \mathbb{H} \to \mathrm{L}(\mathbb{H}; \mathbb{H})$ (see Section 2 [34]).

(A1) There exists a positive constant K_1 such that

$$|\psi(t, \mathbf{u})|_{\mathrm{L_Q}}^2 \le \mathrm{K}_1(1 + |\mathbf{u}|^2), \quad \text{for all } t \in [0, T],\ \mathbf{u} \in \mathbb{H}.$$

(A2) There exists a positive constant M_1 such that

$$|\psi(t,\mathbf{u})-\psi(t,\mathbf{v})|^2_{L_Q}\le M_1|\mathbf{u}-\mathbf{v}|^2,\quad \text{for all } t\in[0,T],\ \mathbf{u},\mathbf{v}\in\mathbb{H}.$$

Poisson random measure is defined as follows: Let $\{\mathbf{L}(t), t\ge 0\}$ be a $\mathbb{H}$-valued Lévy process with jump $\Delta\mathbf{L}(t):=\mathbf{L}(t)-\mathbf{L}(t^-)$ at $t\ge 0$. Then $N([0,t],B)=\#\{s\in[0,t]:\Delta\mathbf{L}(s)\in B\}$ is the Poisson random measure associated with the Lévy process $\{\mathbf{L}(t), t\ge 0\}$, (see Section 2.3, [2]), where $B\in\mathscr{B}(\mathbb{H}\backslash\{0\})$.

Let $\lambda(dx)$ be the σ-finite Lévy measure on $\mathbb{H}$ associated with the Poisson measure $N(dt,dx)$. The compensated Poisson measure is defined as $\widetilde{N}(dt,dx):=N(dt,dx)-\lambda(dx)dt$, that is $E[N(dt,dx)]=\lambda(dx)dt$ (see Section 1 [28]). Let $\mathscr{B}(\mathbb{H})$ be the Borel σ-algebra on $\mathbb{H}$. Then $\mathscr{B}(\mathbb{H}\backslash\{0\})$ is the trace σ-algebra on $\mathbb{H}\backslash\{0\}$. There exists a sequence of Borel measurable sets $\{\mathbb{Z}_m\}_{m=1}^{\infty}\subseteq\mathscr{B}(\mathbb{H}\backslash\{0\})$ such that $0\in(\overline{\mathbb{Z}}_m)^c$, $\lambda(\mathbb{Z}_m)<\infty$ for all $m\in\mathbb{N}$ and $\mathbb{Z}_m\uparrow\mathbb{H}$ as $m\to\infty$.

Note 2.1. For the simplest case, the above limit can be viewed as follows. Let $\lambda(.)$ be a Lévy measure on $\mathbb{R}^n\backslash\{0\}$. Define a sequence $(\epsilon_m, m\in\mathbb{N})$ that decreases monotonically to zero by (see Section 2.6.2,[2])

$$\epsilon_m=\sup\left\{y>0:\int_{0<|x|<y}|x|^2\lambda(\mathrm{d}x)\le\frac{1}{8^m}, x\in\mathbb{R}^n\right\}.$$

Then the sequence of Borel sets $(G_m, m\in\mathbb{N})$ defined by

$$G_m=\{x\in\mathbb{R}^n:|x|>\epsilon_m\},$$

such that $\lambda(G_m)<\infty$ for all $m\in\mathbb{N}$ satisfy $G_m\uparrow\mathbb{R}^n\backslash\{0\}$ when $m\to\infty$.

Let us assume (similar to [49]) that the following conditions hold on jump coefficient function $\phi:\mathbb{H}\times\mathbb{H}\to\mathbb{H}$.

(B1) There exists a positive constant K_2 such that

$$\int_{\mathbb{H}}|\phi(\mathbf{u},x)|^q\lambda(dx)\le K_2(1+|\mathbf{u}|^q),\quad \text{for all } \mathbf{u}\in\mathbb{H},\ q=1,2,4.$$

(B2) There exists a positive constant M_2 such that

$$\int_{\mathbb{H}}|\phi(\mathbf{u},x)-\phi(\mathbf{v},x)|^2\lambda(dx)\le M_2|\mathbf{u}-\mathbf{v}|^2,\quad \text{for all } \mathbf{u},\mathbf{v}\in\mathbb{H}.$$

(B3)

$$\sup_{|\mathbf{u}|\le\tilde{k}}\int_{\mathbb{Z}_n^c}|\phi(\mathbf{u},x)|^2\lambda(dx)\to 0 \text{ as } n\to\infty,\quad \text{for } \mathbf{u}\in\mathbb{H},\ \tilde{k}>0.$$

Now we can express the system (2.1) in abstract form as follows.

$$\mathrm{d}\mathbf{u}(t)+[\nu\mathbf{A}\mathbf{u}(t)+\mathbf{B}(\mathbf{u}(t))]\mathrm{d}t=\mathbf{f}(t)\mathrm{d}t+\psi(t,\mathbf{u}(t))\mathrm{d}W(t)+\int_{\mathbb{H}}\phi(\mathbf{u}(t^-),x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x). \tag{2.14}$$

Note 2.2. Here we note that the last integral make sense as Poisson integral with respect to the compensated Poisson measure $\widetilde{N}(.,.)$, (see [2, Section 4.3.2]). Besides, this Poisson integral is finite due to the quadratic growth rate assumption on the jump noise coefficient $\phi(.,.)$ and the energy estimates and stopping time arguments established in Theorem 2.3 in subsequent section. We use the same notation for noise coefficients and right hand side forcing in (2.1) and their Hodge projection in (2.14).

2.2. *Local Monotonicity Property and A Priori Estimates*

Theorem 2.2 (Local Monotonicity of $(\mathrm{F}(\mathbf{u}), \psi, \phi)$ when $\lambda(.)$ is a finite Lévy measure). *For a given $r > 0$, we consider the following (closed) $\mathbb{L}^p(\mathrm{G}) -$ ball B_r with $p > 2$ in the space $\mathbb{V}$: $\mathrm{B}_r := \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\|_{\mathrm{L}^p(\mathrm{G})} \le r\}$. Define the nonlinear operator F on $\mathbb{V}$ by $\mathrm{F}(\mathbf{u}) := \nu\mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u})$. Then the $(\mathbf{F}(\mathbf{u}), \psi, \phi)$ is monotone in B_r in the following sense:*

$$\begin{aligned}&\langle \mathrm{F}(\mathbf{u}) - \mathrm{F}(\mathbf{v}), \mathbf{w}\rangle - |\psi(t, \mathbf{u}) - \psi(t, \mathbf{v})|^2_{\mathrm{L_Q}}\\&+ \int_{\mathbb{H}} (\phi(\mathbf{u}, x) - \phi(\mathbf{v}, x), \mathbf{w})\lambda(dx)\\&+ \left(\mathrm{C}_{\nu,p} r^{\frac{2p}{p-2}} + \mathrm{M}_1 + \sqrt{\lambda(\mathbb{H})\mathrm{M}_2}\right)|\mathbf{w}|^2 \ge \frac{\nu}{2}\|\mathbf{w}\|^2,\end{aligned} \tag{2.15}$$

for any $\mathbf{u} \in \mathbb{V}$, $\mathbf{v} \in \mathbb{B}_r$ and $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Proof. Let $\mathbf{v}$ and $\mathbf{w}$ be in the spaces $\mathbb{L}^p(\mathrm{G})$ and $\mathbb{V}$, respectively. Then the following estimate holds for $p > 2$.

$$|\langle \mathbf{B}(\mathbf{w}), \mathbf{v}\rangle| \le \mathrm{C}_p \|\mathbf{w}\|^{\frac{p+2}{p}} |\mathbf{w}|^{\frac{p-2}{p}} \|\mathbf{v}\|_{\mathbb{L}^p(\mathrm{G})}. \tag{2.16}$$

(See Section 2, Lemma 3.1, [9]).

From (2.9) and based on the trilinearity of the operator $\mathbf{B}$, we have

$$\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{w}), \mathbf{v}\rangle. \tag{2.17}$$

Now combine (2.16) and (2.17), then apply Young's inequality to get

$$\begin{aligned}|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle| &\le \mathrm{C}_p \|\mathbf{w}\|^{\frac{p+2}{p}} |\mathbf{w}|^{\frac{p-2}{p}} \|\mathbf{v}\|_{\mathbb{L}^p(\mathrm{G})}\\&\le \frac{\nu}{2}\|\mathbf{w}\|^2 + \mathrm{C}_{\nu,p}|\mathbf{w}|^2\|\mathbf{v}\|^{\frac{2p}{p-2}}_{\mathbb{L}^p(\mathrm{G})}.\end{aligned} \tag{2.18}$$

Applying the Cauchy-Schwartz inequality and B2,

$$\begin{aligned}&\int_{\mathbb{H}} |(\phi(\mathbf{u}, x) - \phi(\mathbf{v}, x), \mathbf{w})|\lambda(dx)\\&\le \sqrt{\lambda(\mathbb{H})}\,|\mathbf{w}| \left[\int_{\mathbb{H}} |\phi(\mathbf{u}, x) - \phi(\mathbf{v}, x)|^2\, \lambda(dx)\right]^{\frac{1}{2}}\\&\le \sqrt{\lambda(\mathbb{H})\mathrm{M}_2}|\mathbf{w}|^2.\end{aligned} \tag{2.19}$$

By using the definition of operator F, estimates (2.18), (2.19), $\langle \mathbf{A}\mathbf{w}, \mathbf{w}\rangle = \nu\|\mathbf{w}\|^2$, $\|\mathbf{v}\|_{\mathbb{L}^p(\mathrm{G})} \le r$, condition *A*2, and *B*2, one can complete the proof. □

Let us consider a finite-dimensional Galerkin approximation of the stochastic Navier-Stokes equation with Itô-Lévy noise. Let $\{e_1, e_2, e_3, \ldots, e_i, \ldots\}$ be an orthonormal basis for $\mathbb{H}$ with each $e_i \in \mathrm{D}(\mathbf{A})$. Let $\mathbb{H}_n$ be a finite-dimensional subspace of $\mathbb{H}$ spanned by $\{e_1, e_2, e_3, \ldots, e_n\}$. Let P_n denote the orthonormal projection from $\mathbb{H} \to \mathbb{H}_n$. Let us define $\mathrm{W}_n = \mathrm{P}_n\mathrm{W}$, $\psi_n = \mathrm{P}_n\psi$ and $\phi_n = \mathrm{P}_n\phi$. We consider the finite-dimensional stochastic Navier-Stokes equation in variational form for $\mathbf{u}_n(t)$:

$$
\begin{aligned}
(\mathbf{u}_n(t), \mathbf{v}) &= (\mathbf{u}_n(0), \mathbf{v}) - \int_0^t (\mathbf{A}\mathbf{u}_n(s), \mathbf{v})\mathrm{d}s - \int_0^t (\mathbf{B}_n(\mathbf{u}_n(s)), \mathbf{v})\mathrm{d}s \\
&\quad + \int_0^t (\mathbf{f}_n(s), \mathbf{v})\mathrm{d}s + \int_0^t (\psi_n(s, \mathbf{u}_n(s))\mathrm{d}\mathrm{W}_n(s), \mathbf{v}) \\
&\quad + \int_0^t \int_{\mathbb{Z}_n} (\phi_n(\mathbf{u}_n(s^-), x), \mathbf{v})\widetilde{N}(\mathrm{d}s, \mathrm{d}x), \qquad (2.20)
\end{aligned}
$$

with $\mathbf{u}_0^n = \mathrm{P}_n\mathbf{u}_0$ for each $\mathbf{v} \in \mathbb{H}_n$.

Note 2.3. In the finite-dimensional system (2.20), we take compensated Poisson integral with the Lévy measure $\lambda(\mathbb{Z}_n) < \infty$ and $\mathbb{Z}_n \uparrow \mathbb{H}$.

We follow the techniques in [37] (see also [1]) to obtain the following a priori estimate results.

Theorem 2.3. *Let $\mathbf{u}_n(t)$ be a adapted process in $\mathbb{D}(0, T; \mathbb{H}_n)$ that satisfies (2.20). Under the assumptions $\psi(.,.)$ and $\phi(.,.)$ (i.e., A2 and B1), we have following estimates.*

1. *Let $\mathbf{f} \in \mathbb{L}^2(\Omega; \mathbb{L}^2(0, T; \mathbb{V}'))$ and $\mathrm{E}|\mathbf{u}_0|^2 < \infty$, then for all $0 \le t \le T$,*

$$
\begin{aligned}
&\mathrm{E}\left(\sup_{0\le s\le t} |\mathbf{u}_n(s)|^2 + \nu \int_0^t \|\mathbf{u}_n(s)\|^2 \mathrm{d}s \right) \\
&\quad \le C\left(\mathrm{E}|\mathbf{u}_0|^2, \int_0^T \mathrm{E}\|\mathbf{f}(s)\|_{\mathbb{V}'}^2 \mathrm{d}s, \nu, T, \mathrm{K}_1, \mathrm{K}_2 \right). \qquad (2.21)
\end{aligned}
$$

2. *Let $\mathbf{f} \in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{V}'))$ and $\mathrm{E}|\mathbf{u}_0|^4 < \infty$, then for all $0 \le t \le T$,*

$$
\begin{aligned}
&\mathrm{E}\left(\sup_{0\le s\le t} |\mathbf{u}_n(s)|^4 + 2\nu \int_0^t |\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2 \mathrm{d}s \right) \\
&\quad \le C\left\{ \mathrm{E}|\mathbf{u}_0|^4, \int_0^T \mathrm{E}\|\mathbf{f}(s)\|_{\mathbb{V}'}^4 \mathrm{d}s, \nu, T, \mathrm{K}_1, \mathrm{K}_2 \right\}. \qquad (2.22)
\end{aligned}
$$

Proof. We start with the finite-dimensional system of $\mathbb{H}_n$ valued stochastic ordinary differential equation (2.20):

$$
\begin{aligned}
\mathrm{d}\mathbf{u}_n(t) &= -[\mathbf{A}\mathbf{u}_n(t) + \mathbf{B}_n(\mathbf{u}_n(t))]\mathrm{d}t + \mathbf{f}_n(t)\mathrm{d}t \\
&\quad + \psi_n(t, \mathbf{u}_n(t))\mathrm{d}\mathrm{W}_n(t) + \int_{\mathbb{Z}_n} (\phi_n(\mathbf{u}_n(t^-), x)\widetilde{N}(\mathrm{d}t, \mathrm{d}x) \qquad (2.23)
\end{aligned}
$$

For any fixed $N \ge 1$ and $n \ge 1$, define $\tau_N^n := \inf\{t : |\mathbf{u}_n(t)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 \mathrm{d}s \ge N\}$.

By means of Itô's Lemma (Theorem 4.4.10 of [2] or Section 4 of [33] and also find Itô formula in [6, 14, 16, 31]) and (2.23), we obtain

$$\begin{aligned}|\mathbf{u}_n(t\wedge\tau_N^n)|^2 &= |\mathbf{u}_n(0)|^2 - 2\nu\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)\|^2\mathrm{d}s + 2\int_0^{t\wedge\tau_N^n}\langle\mathbf{f}_n(s),\mathbf{u}_n(s)\rangle\mathrm{d}s\\&\quad + 2\int_0^{t\wedge\tau_N^n}(\psi_n(s,\mathbf{u}_n(s))\mathrm{dW}_n(s),\mathbf{u}_n(s))\\&\quad + 2\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_n}((\phi_n(\mathbf{u}_n(s^-),x),\mathbf{u}_n(s^-))\widetilde{N}(\mathrm{d}s,\mathrm{d}x)\\&\quad + \int_0^{t\wedge\tau_N^n}(\psi_n(s,\mathbf{u}_n(s))\mathrm{dW}_n(s),\psi_n(t,\mathbf{u}_n(s))\mathrm{dW}_n(s))\\&\quad + \sum_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_n(s^-)|^2,\end{aligned}\tag{2.24}$$

where the jumps $|\mathbf{u}_n(s)-\mathbf{u}_n(s^-)|$ are in $\mathbb{Z}_n$. By noting $2\langle\mathbf{f}_n(t),\mathbf{u}_n(t)\rangle\le\nu\|\mathbf{u}_n(t)\|^2+\frac{1}{\nu}\|\mathbf{f}_n(t)\|^2_{\mathbb{V}'}$ and taking the supremum of both sides of (2.24) before applying expectation

$$\begin{aligned}&\mathrm{E}\left(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\right)+\nu\int_0^{t\wedge\tau_N^n}\mathrm{E}\|\mathbf{u}_n(s)\|^2\mathrm{d}s\le\mathrm{E}|\mathbf{u}_n(0)|^2\\&\quad+\frac{1}{\nu}\int_0^{T\wedge\tau_N^n}\mathrm{E}\|\mathbf{f}_n(s)\|^2_{\mathbb{V}'}\mathrm{d}s+2\mathrm{E}\left(\sup_{0\le s\le t\wedge\tau_N^n}\left|\int_0^s(\psi_n(\acute{s},\mathbf{u}_n(\acute{s}))\mathrm{dW}_n(\acute{s}),\mathbf{u}_n(\acute{s}))\right|\right)\\&\quad+2\mathrm{E}\left(\sup_{0\le s\le t\wedge\tau_N^n}\left|\int_0^s\int_{\mathrm{Z_n}}(\phi_n(\mathbf{u}_n(\acute{s}-),x),\mathbf{u}_n(\acute{s}-))\widetilde{N}(\mathrm{d}\acute{s},\mathrm{d}x)\right|\right)\\&\quad+\mathrm{E}\int_0^{t\wedge\tau_N^n}\mathrm{Tr}(\psi_n(s,\mathbf{u}_n(s))\mathrm{Q}\psi_n^*(s,\mathbf{u}_n(s)))\mathrm{d}s\\&\quad+\mathrm{E}\sum_{0\le s\le t\wedge\tau_N^n}|\phi_n(\mathbf{u}_n(s^-),\Delta\mathbf{u}_n(s))|^2.\end{aligned}\tag{2.25}$$

Applying the Burkholder-Davis-Gundy inequality, Cauchy-Schwartz inequality, and condition (A1) for first martingale term in the right-hand side of (2.25):

$$\begin{aligned}&2\mathrm{E}\left(\sup_{0\le s\le t\wedge\tau_N^n}\left|\int_0^s(\psi_n(\acute{s},\mathbf{u}_n(\acute{s}))\mathrm{dW}_n(\acute{s}),\mathbf{u}_n(\acute{s}))\right|\right)\\&\quad\le 2\sqrt{2}\mathrm{K}_1\mathrm{E}\left(\int_0^{t\wedge\tau_N^n}(1+|\mathbf{u}_n(s)|^2)|\mathbf{u}_n(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\\&\quad\le\frac{\epsilon}{2}\mathrm{E}\left(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\right)+\frac{4\mathrm{K_1}^2}{\epsilon}\mathrm{E}\int_0^{t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\mathrm{d}s+\frac{4\mathrm{K_1}^2}{\epsilon}T.\end{aligned}\tag{2.26}$$

Similarly, we can estimate the martingale term involved with the compensated Poisson process as follows:

$$2\mathrm{E}\left(\sup_{0\le s\le t\wedge\tau_N^n}\left|\int_0^s\int_{\mathrm{Z_n}}(\phi_n(\mathbf{u}_n(\acute{s}-),x),\mathbf{u}_n(\acute{s}-))\widetilde{N}(\mathrm{d}\acute{s},\mathrm{d}x)\right|\right)$$

$$\le 2\sqrt{2}\mathrm{E}\bigg(\int_0^{t\wedge\tau_N^n}\int_{\mathrm{Z_n}}|(\phi_n(\mathbf{u}_n(s^-),x),\mathbf{u}_n(s^-))|^2 N(\mathrm{d}\acute{s},\mathrm{d}x)\bigg)^{\frac{1}{2}}$$

$$\le 2\sqrt{2}\mathrm{E}\bigg(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\bigg)^{\frac{1}{2}}\bigg(\int_0^{t\wedge\tau_N^n}\int_{\mathrm{Z_n}}|\phi_n(\mathbf{u}_n(s^-),x)|^2 N(\mathrm{d}\acute{s},\mathrm{d}x)\bigg)^{\frac{1}{2}}$$

$$\le \frac{\epsilon}{2}\mathrm{E}\bigg(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\bigg)+\frac{4}{\epsilon}\int_0^{t\wedge\tau_N^n}\int_{\mathrm{Z_n}}\mathrm{E}|\phi_n(\mathbf{u}_n(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s$$

$$\le \frac{\epsilon}{2}\mathrm{E}\bigg(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\bigg)+\frac{4\mathrm{K}_2}{\epsilon}\mathrm{E}\int_0^{t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\mathrm{d}s+\frac{4\mathrm{K}_2}{\epsilon}T. \tag{2.27}$$

Let us estimate the last term in the right-hand side of (2.25);

$$\mathrm{E}\sum_{0\le s\le t\wedge\tau_N^n}|\phi_n(\mathbf{u}_n(s^-),\Delta\mathbf{u}_n(s))|^2$$

$$=\mathrm{E}\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|^2 N(\mathrm{d}s,\mathrm{d}x)$$

$$=\mathrm{E}\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_n}|(\phi_n(\mathbf{u}_n(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s. \tag{2.28}$$

Replacing (2.26), (2.27), (2.28) in (2.25) and applying conditions A1 and B1,

$$\mathrm{E}\bigg(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\bigg)+v\mathrm{E}\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)\|^2\mathrm{d}s\le \mathrm{E}|\mathbf{u}_n(0)|^2$$

$$+\frac{1}{v}\mathrm{E}\int_0^{T\wedge\tau_N^n}\|\mathbf{f}_n(s)\|^2_{\mathbb{V}'}\mathrm{d}s+\epsilon\mathrm{E}\bigg(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\bigg)$$

$$+\frac{4{\mathrm{K}_1}^2}{\epsilon}\mathrm{E}\int_0^{t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\mathrm{d}s+\frac{4\mathrm{K}_2}{\epsilon}\mathrm{E}\int_0^{t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\mathrm{d}s$$

$$+\frac{4({\mathrm{K}_1}^2+\mathrm{K}_2)}{\epsilon}T+\mathrm{E}\int_0^{t\wedge\tau_N^n}\mathrm{Tr}(\psi_n(s,\mathbf{u}_n(s))\mathrm{Q}\psi_n^*(s,\mathbf{u}_n(s)))\mathrm{d}s$$

$$+\mathrm{E}\int_0^{t\wedge\tau_N^n}\int_{\mathrm{Z_n}}|\phi_n(\mathbf{u}_n(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s\le \mathrm{E}|\mathbf{u}_n(0)|^2$$

$$+\frac{1}{v}\mathrm{E}\int_0^{T\wedge\tau_N^n}\|\mathbf{f}_n(s)\|^2_{\mathbb{V}'}\mathrm{d}s+\epsilon\mathrm{E}\bigg(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\bigg)$$

$$+\mathrm{C}_1\mathrm{E}\int_0^{t\wedge\tau_N^n}\bigg(\sup_{\acute{s}\le s}|\mathbf{u}_n(\acute{s})|^2\bigg)\mathrm{d}s+\mathrm{C}_1T. \tag{2.29}$$

Thus, we have

$$(1-\epsilon)\mathrm{E}\bigg(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\bigg)+v\mathrm{E}\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)\|^2\mathrm{d}s\le \mathrm{E}|\mathbf{u}_0|^2$$

$$+\frac{1}{v}\mathrm{E}\int_0^{T}\|\mathbf{f}_n(s)\|^2_{\mathbb{V}'}\mathrm{d}s+\mathrm{C}_1\mathrm{E}\int_0^{t\wedge\tau_N^n}\bigg(\sup_{\acute{s}\le s}|\mathbf{u}_n(\acute{s})|^2\bigg)\mathrm{d}s+\mathrm{C}_1T. \tag{2.30}$$

Applying Gronwall's inequality to (2.30) with $\epsilon = \frac{1}{2}$, we get

$$\begin{aligned} &\mathrm{E}\left(\sup_{0\le s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)|^2\right) + \nu\mathrm{E}\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)\|^2\mathrm{d}s \\ &\quad\le 2\left(\mathrm{E}|\mathbf{u}_0|^2 + \frac{1}{\nu}\mathrm{E}\int_0^T\|\mathbf{f}_n(s)\|^2_{\mathbb{V}'}\mathrm{d}s + \mathrm{C}_1T\right)\exp\{2\mathrm{C}_1T\}. \end{aligned} \tag{2.31}$$

Now define

$$\Omega_N := \left\{\omega\in\Omega : |\mathbf{u}_n(t)|^2 + \int_0^t\|\mathbf{u}_n(s)\|^2\mathrm{d}s < N\right\}, \tag{2.32}$$

Then we have

$$\begin{aligned} &\int_{\Omega_N}\left(|\mathbf{u}_n(t)|^2 + \int_0^t\|\mathbf{u}_n(s)\|^2\mathrm{d}s\right)\mathrm{P}(\mathrm{d}\omega) \\ &\quad + \int_{\Omega\backslash\Omega_N}\left(|\mathbf{u}_n(t)|^2 + \int_0^t\|\mathbf{u}_n(s)\|^2\mathrm{d}s\right)\mathrm{P}(\mathrm{d}\omega) \le \mathrm{C}_2. \end{aligned} \tag{2.33}$$

Then by dropping the first integral and using the fact that $|\mathbf{u}_n(t)|^2 + \int_0^t\|\mathbf{u}_n(s)\|^2\mathrm{d}s \ge N$ in $\Omega\backslash\Omega_N$,

$$\mathrm{P}\{\Omega\backslash\Omega_N\} \le \frac{\mathrm{C}_2}{N}. \tag{2.34}$$

Note also that

$$\mathrm{P}\{\omega\in\Omega : \tau_N^n < t\} = \mathrm{P}\{\Omega\backslash\Omega_N\} \le \frac{\mathrm{C}_2}{N}, \quad \text{for any } t\le T. \tag{2.35}$$

Hence, $\limsup_{N\to\infty}\mathrm{P}\{\omega\in\Omega : \tau_N^n < t\} = 0$. Therefore, $\tau_N^n \to t$ as $N\to\infty$. Then taking the limit of (2.31) as $N\to\infty$, we get (2.21).

Now we will prove the second estimate.

Define $\theta_N^n := \inf\{t : |\mathbf{u}_n(t)|^4 + \int_0^t|\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s \ge N\}$, then apply Itô's Lemma (Theorem 4.4.10 of [2] or Section 4 of [33]) to the function $\mathbf{u}\to|\mathbf{u}|^4$ to get,

$$\begin{aligned} |\mathbf{u}_n(t\wedge\theta_N^n)|^4 &= |\mathbf{u}_n(0)|^4 - 4\nu\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s \\ &\quad + 4\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\langle\mathbf{f}_n(s),\mathbf{u}_n(s)\rangle\mathrm{d}s \\ &\quad + 4\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2(\psi_n(s,\mathbf{u}_n(s))\mathrm{dW}_n(s),\mathbf{u}_n(s)) \\ &\quad + 4\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\int_{\mathbb{Z}_n}((\phi_n(\mathbf{u}_n(s^-),x),\mathbf{u}_n(s^-))\widetilde{N}(\mathrm{d}s,\mathrm{d}x) \\ &\quad + 6\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\mathrm{Tr}(\psi_n(s,\mathbf{u}_n(s))\mathrm{Q}\psi_n^*(s,\mathbf{u}_n(s)))\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0\le s\le t\wedge\theta_N^n} \Big[|\mathbf{u}_n(s)-\mathbf{u}_n(s^-)|^4 + 4|\mathbf{u}_n(s)|^2(\mathbf{u}_n(s),\mathbf{u}_n(s^-)) \\
&- 2|\mathbf{u}_n(s)|^2|\mathbf{u}_n(s^-)|^2 - 4(\mathbf{u}_n(s),\mathbf{u}_n(s^-))^2 + 2|\mathbf{u}_n(s^-)|^4 \Big], \qquad (2.36)
\end{aligned}
$$

where the jumps $|\mathbf{u}_n(s)-\mathbf{u}_n(s^-)|$ are in $\mathbb{Z}_n$. Now apply Young's inequality to the term $\langle \mathbf{f}_n(t), \mathbf{u}_n(t)\rangle$, condition A1 to the sixth term in right-hand side of (2.36), and taking supremum of both sides before applying expectation, we get

$$
\begin{aligned}
&\mathrm{E}\Big(\sup_{0\le s\le t\wedge\theta_N^n} |\mathbf{u}_n(s)|^4\Big) + 2\nu\mathrm{E}\int_0^{t\wedge\theta_N^n} |\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s \le \mathrm{E}|\mathbf{u}_n(0)|^4 \\
&+ \frac{2}{\nu}\mathrm{E}\int_0^{t\wedge\theta_N^n} |\mathbf{u}_n(s)|^2\|\mathbf{f}_n(s)\|_{\mathbb{V}'}^2\mathrm{d}s \\
&+ 4\mathrm{E}\Big(\sup_{0\le s\le t\wedge\theta_N^n}\Big|\int_0^s |\mathbf{u}_n(\acute{s})|^2(\psi_n(\acute{s},\mathbf{u}_n(\acute{s}))\mathrm{d}\mathrm{W}_n(\acute{s}),\mathbf{u}_n(\acute{s}))\Big|\Big) \\
&+ 4\mathrm{E}\Big(\sup_{0\le s\le t\wedge\theta_N^n}\Big|\int_0^s |\mathbf{u}_n(\acute{s})|^2\int_{\mathrm{Z_n}}(\phi_n(\mathbf{u}_n(\acute{s}-),x),\mathbf{u}_n(\acute{s}-))\widetilde{N}(\mathrm{d}\acute{s},\mathrm{d}x)\Big|\Big) \\
&+ 6\mathrm{K}_1\mathrm{E}\int_0^{t\wedge\theta_N^n} |\mathbf{u}_n(s)|^2(1+|\mathbf{u}_n(s)|^2)\mathrm{d}s \\
&+ \mathrm{E}\sum_{0\le s\le t\wedge\theta_N^n}\Big[|\Delta\mathbf{u}_n(s)|^4 + 4|\mathbf{u}_n(s)|^2|\mathbf{u}_n(s^-)||\Delta\mathbf{u}_n(s)| \\
&+ 2|\mathbf{u}_n(s^-)|^3|\Delta\mathbf{u}_n(s)| + 2|\mathbf{u}_n(s^-)|^2|\mathbf{u}_n(s)||\Delta\mathbf{u}_n(s)|\Big], \qquad (2.37)
\end{aligned}
$$

where $|\Delta\mathbf{u}_n(s)| = |\mathbf{u}_n(s)-\mathbf{u}_n(s^-)|$. The Cauchy-Schwartz inequality, and triangle inequality have been applied for the terms associated with jumps in (2.36).

The second term in the right-hand side of (2.37) is estimated by means of Young's inequality as follows.

$$
\begin{aligned}
\frac{2}{\nu}\mathrm{E}\int_0^{t\wedge\theta_N^n} |\mathbf{u}_n(s)|^2\|\mathbf{f}_n(s)\|_{\mathbb{V}'}^2\mathrm{d}s &\le \frac{\epsilon}{4}\mathrm{E}\Big(\sup_{0\le s\le t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\Big) \\
&+ \frac{4}{\nu^2\epsilon}\mathrm{E}\int_0^{T\wedge\theta_N^n}\|\mathbf{f}_n(s)\|_{\mathbb{V}'}^4\mathrm{d}s + \frac{4}{\nu^2\epsilon}T. \qquad (2.38)
\end{aligned}
$$

By using Burkholder-Davis-Gundy inequality, Cauchy-Schwartz inequality, and condition A1,

$$
\begin{aligned}
&4\mathrm{E}\Big(\sup_{0\le s\le t\wedge\theta_N^n}\Big|\int_0^s |\mathbf{u}_n(s)|^2(\psi_n(\acute{s},\mathbf{u}_n(\acute{s}))\mathrm{d}\mathrm{W}_n(\acute{s}),\mathbf{u}_n(\acute{s}))\Big|\Big) \\
&\le \frac{\epsilon}{4}\mathrm{E}\Big(\sup_{0\le s\le t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\Big) \\
&+ \frac{16}{\epsilon}\mathrm{E}\Big(\sup_{0\le s\le t\wedge\theta_N^n}\Big|\int_0^s(\psi_n(\acute{s},\mathbf{u}_n(\acute{s}))\mathrm{d}\mathrm{W}_n(\acute{s}),\mathbf{u}_n(\acute{s}))\Big|\Big)^2
\end{aligned}
$$

$$\leq \frac{\epsilon}{4}\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4)\bigg)+\frac{16\sqrt{2}\mathrm{K}_1}{\epsilon}\mathrm{E}\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2(1+|\mathbf{u}_n(s)|^2)\mathrm{d}s$$

$$\leq \frac{\epsilon}{4}\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4)\bigg)+\frac{24\sqrt{2}\mathrm{K}_1}{\epsilon}\mathrm{E}\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\mathrm{d}s+\frac{16\sqrt{2}\mathrm{K}_1}{\epsilon}T. \quad (2.39)$$

We can estimate the martingale term associated with compensated Poisson process by similar procedure using Condition B1;

$$4\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}\bigg|\int_0^s|\mathbf{u}_n(\acute{s})|^2\int_{\mathrm{Z_n}}(\phi_n(\mathbf{u}_n(\acute{s}-),x),\mathbf{u}_n(\acute{s}-))\widetilde{N}(\mathrm{d}\acute{s},\mathrm{d}x)\bigg|\bigg)$$

$$\leq \frac{\epsilon}{4}\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4)\bigg)+\frac{16\sqrt{2}\mathrm{K}_2}{\epsilon}\mathrm{E}\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2(1+|\mathbf{u}_n(s)|^2)\mathrm{d}s$$

$$\leq \frac{\epsilon}{4}\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4)\bigg)+\frac{24\sqrt{2}\mathrm{K}_2}{\epsilon}\mathrm{E}\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\mathrm{d}s+\frac{16\sqrt{2}\mathrm{K}_2}{\epsilon}T. \quad (2.40)$$

Let us estimate the term involving jumps in (2.37).

$$\begin{aligned}
&\mathrm{E}\sum_{0\leq s\leq t\wedge\theta_N^n}\Big[|\Delta\mathbf{u}_n(s)|^4+4|\mathbf{u}_n(s)|^2|\mathbf{u}_n(s^-)||\Delta\mathbf{u}_n(s)| \\
&\qquad +2|\mathbf{u}_n(s^-)|^3|\Delta\mathbf{u}_n(s)|+2|\mathbf{u}_n(s^-)|^2|\mathbf{u}_n(s)||\Delta\mathbf{u}_n(s)|\Big] \\
&\quad\leq \mathrm{E}\sum_{0\leq s\leq t\wedge\theta_N^n}|\Delta\mathbf{u}_n(s)|^4 \\
&\qquad +8\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\bigg)\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s^-)|\bigg)\sum_{0\leq s\leq t\wedge\theta_N^n}|\Delta\mathbf{u}_n(s)| \\
&\quad\leq \mathrm{E}\sum_{0\leq s\leq t\wedge\theta_N^n}|(\phi_n(\mathbf{u}_n(s^-),\Delta\mathbf{u}_n(s))|^4+8\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\bigg) \\
&\qquad\times\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|\bigg)\sum_{0\leq s\leq t\wedge\theta_N^n}|(\phi_n(\mathbf{u}_n(s^-),\Delta\mathbf{u}_n(s))| \\
&\quad\leq \mathrm{E}\int_0^{t\wedge\theta_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|^4\,N(\mathrm{d}s,\mathrm{d}x)+8\bigg[\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\bigg)^2\bigg]^{\frac{1}{2}} \\
&\qquad\times\bigg[\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|\bigg)^4\bigg]^{\frac{1}{4}}\bigg[\mathrm{E}\bigg(\int_0^{t\wedge\theta_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|\,N(\mathrm{d}s,\mathrm{d}x)\bigg)^4\bigg]^{\frac{1}{4}} \\
&\quad\leq \mathrm{E}\int_0^{t\wedge\theta_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|^4\,N(\mathrm{d}s,\mathrm{d}x)+\frac{\epsilon}{4}\mathrm{E}\bigg(\sup_{0\leq s\leq t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\bigg) \\
&\qquad+8\left(\frac{24}{\epsilon}\right)^3\mathrm{E}\bigg(\int_0^{t\wedge\theta_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|\,\tilde{N}(\mathrm{d}s,\mathrm{d}x)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t\wedge\theta_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|\,\lambda(\mathrm{d}x)\mathrm{d}s\Bigg)^4 \\
&\le \mathrm{E}\int_0^{t\wedge\theta_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|^4\,\lambda(\mathrm{d}x)\mathrm{d}s+\frac{\epsilon}{4}\mathrm{E}\Bigg(\sup_{0\le s\le t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\Bigg) \\
&\quad+8\breve{C}\left(\frac{24}{\epsilon}\right)^3\mathrm{E}\Bigg(\int_0^{t\wedge\theta_N^n}\int_{\mathbb{Z}_n}|\phi_n(\mathbf{u}_n(s^-),x)|^2\,\lambda(\mathrm{d}x)\mathrm{d}s\Bigg)^2 \\
&\quad+8\breve{C}\mathrm{K}_2{}^2\left(\frac{24}{\epsilon}\right)^3\mathrm{E}\Bigg(\int_0^{t\wedge\theta_N^n}(1+|\mathbf{u}_n(s)|^2)\mathrm{d}s\Bigg)^2 \\
&\quad+8\tilde{C}\mathrm{K}_2{}^4\left(\frac{24}{\epsilon}\right)^3\mathrm{E}\Bigg(\int_0^{t\wedge\theta_N^n}(1+|\mathbf{u}_n(s)|)\mathrm{d}s\Bigg)^4 \\
&\le \frac{\epsilon}{4}\mathrm{E}\Bigg(\sup_{0\le s\le t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\Bigg)+\left[16T\mathrm{K}_2{}^2\left(\frac{24}{\epsilon}\right)^3(\breve{C}+2T^2\mathrm{K}_2{}^2\tilde{C})+\mathrm{K}_2\right] \\
&\quad\times\mathrm{E}\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\mathrm{d}s+\left[16\mathrm{K}_2{}^2\left(\frac{24}{\epsilon}\right)^3(\breve{C}+2T^2\mathrm{K}_2{}^2\tilde{C})+\mathrm{K}_2\right]T.
\end{aligned}
$$

(2.41) holds due to Cauchy-Schwartz inequality, Young's inequality, and condition B1. Combining (2.37), (2.38), (2.39), (2.40), and (2.41), reorganizing terms,

$$
\begin{aligned}
&(1-\epsilon)\mathrm{E}\Bigg(\sup_{0\le s\le t\wedge\theta_N^n}|\mathbf{u}_n(s)|^4\Bigg)+2\nu\mathrm{E}\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s\le\mathrm{E}|\mathbf{u}_0|^4 \\
&\quad+\frac{4}{\nu^2\epsilon}\mathrm{E}\int_0^T\|\mathbf{f}_n(s)\|_{\mathbb{V}'}^4\mathrm{d}s+\mathrm{C}_3\mathrm{E}\int_0^{t\wedge\theta_N^n}\left(\sup_{\acute{s}\le s}|\mathbf{u}_n(\acute{s})|^4\right)\mathrm{d}s+\mathrm{C}_4T. \qquad (2.41)
\end{aligned}
$$

By means of Gronwall's Lemma with $\epsilon=\frac{1}{2}$, we get

$$
\begin{aligned}
&\mathrm{E}\Bigg(\sup_{0\le s\le t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\Bigg)+2\nu\mathrm{E}\int_0^{t\wedge\theta_N^n}|\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s \\
&\quad\le 2\Bigg(\mathrm{E}|\mathbf{u}_0|^2+\frac{8}{\nu^2}\mathrm{E}\int_0^T\|\mathbf{f}_n(s)\|_{\mathbb{V}'}^4\mathrm{d}s+\mathrm{C}_3T\Bigg)\exp\{2\mathrm{C}_4T\}. \qquad (2.42)
\end{aligned}
$$

Define

$$
\hat{\Omega}_N:=\left\{\omega\in\Omega:|\mathbf{u}_n(t)|^4+\int_0^t|\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s<N\right\}, \qquad (2.43)
$$

then

$$
\begin{aligned}
&\int_{\hat{\Omega}_N}\left(|\mathbf{u}_n(t)|^4+\int_0^t|\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s\right)\mathrm{P}(\mathrm{d}\omega) \\
&\quad+\int_{\Omega\backslash\hat{\Omega}_N}\left(|\mathbf{u}_n(t)|^4+\int_0^t|\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s\right)\mathrm{P}(\mathrm{d}\omega)\le\acute{\mathrm{C}}. \qquad (2.44)
\end{aligned}
$$

Then by dropping the first integral and using the fact that, $|\mathbf{u}_n(t)|^4 + \int_0^t |\mathbf{u}_n(s)|^2\|\mathbf{u}_n(s)\|^2 \mathrm{d}s \geq N$ in $\Omega\backslash\hat{\Omega}_N$. We obtain,

$$\mathrm{P}\left\{\Omega\backslash\hat{\Omega}_N\right\} \leq \frac{\acute{\mathrm{C}}}{N}. \tag{2.45}$$

Note also that

$$\mathrm{P}\left\{\omega \in \Omega : \tau_N^n < t\right\} = \mathrm{P}\left\{\Omega\backslash\hat{\Omega}_N\right\} \leq \frac{\mathrm{C}_4}{N}, \quad \text{for any } t \leq T. \tag{2.46}$$

Hence, $\limsup_{N\to\infty} \mathrm{P}\{\omega \in \Omega : \theta_N^n < t\} = 0$. Therefore, $\theta_N^n \to t$ as $N \to \infty$. Therefore, by taking $N \to \infty$, we will get the estimate (2.22). □

2.3. *Existence and Uniqueness*

Definition 2.2. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathrm{P})$ be a complete probability space equipped with a filtration $\mathscr{F}_t$. Let $\mathbf{u}(t)$ be a $\mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H})) \cap \mathbb{L}^2(\Omega \times [0, T]; \mathbb{V})$-valued cádlág (with respect to $\mathbb{D}(0, T; \mathbb{H})$) adapted process. Suppose that $\mathbf{u}(t)$ satisfies stochastic Navier-Stokes equation in weak sense almost surely:

$$\begin{aligned}(\mathbf{u}(t), \mathbf{v}) = (\mathbf{u}(0), \mathbf{v}) &- \int_0^t \langle \mathbf{A}\mathbf{u}(s), \mathbf{v}\rangle\, \mathrm{d}s - \int_0^t \langle \mathbf{B}(\mathbf{u}(s)), \mathbf{v}\rangle\, \mathrm{d}s \\ &+ \int_0^t (\mathbf{f}(s), \mathbf{v})\mathrm{d}s + \int_0^t (\psi(s, \mathbf{u}(s))\mathrm{d}\mathrm{W}(t), \mathbf{v}) \\ &+ \int_0^t \int_{\mathbb{H}} (\phi(\mathbf{u}(s^-), x), \mathbf{v})\widetilde{N}(\mathrm{d}s, \mathrm{d}x) \text{ a.s.}, \quad \text{for all } \mathbf{v} \in \mathbb{V},\end{aligned} \tag{2.47}$$

and also the energy inequalities in Theorem 2.3. Then $\mathbf{u}(t)$ is called a strong pathwise solution of (2.14).

2.3.1. Lévy Measure $\lambda(.)$ is Finite. In this section, we first prove existence and uniqueness of solution of (2.14) associated with the finite Lévy measure (i.e. $\lambda(\mathbb{H}) < \infty$) using local monotonicity method and generalization of the Minty-Browder technique. The major advantage of this technique is the complete elimination of compact embedding theorem that is only available for bounded domains.

Some of the earlier works on monotonicity method for stochastic partial differential equations are [19, 25, 27, 29], and [43], who proved the existence and uniqueness of strong and martingale solutions for a wide class of stochastic evolution equations. Menaldi and Sritharan [26] generalized this method for local monotonicity so that it is applicable for stochastic Navier-Stokes equations.

Since $\lambda(\mathbb{H}) < \infty$, we can organize the jump times of $N(\mathrm{d}s, \mathrm{d}x)$ as follows (see [49]):

$$\mu_1(\omega) < \mu_2(\omega) < \mu_3(\omega) < \cdots.$$

Since $\int_0^t \int_{\mathbb{H}} \phi(\mathbf{u}(t^-), x)N(\mathrm{d}t, \mathrm{d}x) = 0$ on time interval $[0, \mu_1(\omega))$, the stochastic Navier-Stokes equation (2.14) reduces to following form.

$$\begin{aligned}&\mathrm{d}\mathbf{u}(t) + [\nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t))]\mathrm{d}t \\ &\quad = \mathbf{f}(t)\mathrm{d}t + \psi(t, \mathbf{u}(t))\mathrm{d}\mathrm{W}(t) - \int_{\mathbb{H}} \phi(\mathbf{u}(t), x)\lambda(\mathrm{d}x)\mathrm{d}t.\end{aligned} \tag{2.48}$$

The finite-dimensional stochastic Navier-Stokes equation in variational formulation on the interval $[0, \mu_1(\omega))$ can be represented as follows:

$$\begin{aligned}(\mathbf{u}_n(t), \mathbf{v}) = (\mathbf{u}_n(0), \mathbf{v}) + \int_0^t (\mathbf{f}_n(s), \mathbf{v})\mathrm{d}s \\ - \int_0^t (\mathbf{A}\mathbf{u}_n(s) + \mathbf{B}_n(\mathbf{u}_n(s)), \mathbf{v})\mathrm{d}s - \int_{\mathbb{Z}_n} (\phi_n(\mathbf{u}_n(s), x)\lambda(\mathrm{d}x), \mathbf{v})\mathrm{d}s \\ + \int_0^t (\psi_n(s, \mathbf{u}_n(s))\mathrm{d}W_n(s), \mathbf{v}),\end{aligned} \tag{2.49}$$

with $\mathbf{u}_0^n = P_n\mathbf{u}_0$ for each $\mathbf{v} \in \mathbb{H}_n$.

Theorem 2.4. *Let* $\mathbf{f} \in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{V}'))$ *and the initial value* $\mathbf{u}_0$ *satisfies* $E|\mathbf{u}_0|^4 < \infty$. *Suppose that the conditions associated with* $\psi(.,.)$ *and* $\phi(.,.)$ (*i.e., A*1-*A*2 *and B*1-*B*3) *hold. Then the model* (2.48) *has a unique adapted solution* $\mathbf{u}(t, \omega, x)$ *on* $[0, \mu_1(\omega))$ *in* $\mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H}) \cap \mathbb{C}(0, T; \mathbb{H})) \cap \mathbb{L}^2(\Omega \times [0, T); \mathbb{V})$.

Proof. Firstly, we prove existence results.

1. Weak convergent sub-sequences.

According to Banach-Alaoglu theorem and Theorem 2.3, we can extract sub-sequences from Galerkin approximation $\{\mathbf{u}_n(t)\}$ and each sub-sequence have following limits.

- $\mathbf{u}_n(.) \to \mathbf{u}(.)$ weak star in $\mathbb{L}^4(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H})) \cap \mathbb{L}^2(\Omega \times [0, T]; \mathbb{V})$,
- $\widetilde{F}(\mathbf{u}_n(.)) \to \widetilde{F}_0(.)$ weakly in $\mathbb{L}^2(\Omega \times [0, T]; \mathbb{V}')$, where the operator $\widetilde{F}(.) = F(.) - \mathbf{f}$,
  The following two statements hold since $\psi(.,.)$ and $\phi(.,.)$ satisfy condition A1 and B1, respectively, and first part of the a priori estimates.
- $\psi_n(t, \mathbf{u}_n(.)) \to S(.)$ weakly in $\mathbb{L}^2(\Omega \times [0, T]; L_Q)$ and
- $\int_{\mathbb{H}} \phi_n(\mathbf{u}_n(s), x)\lambda(\mathrm{d}x) \to J(.)$ weakly in $\mathbb{L}^2(\Omega \times [0, T]; \mathbb{V}')$.

Then $\mathbf{u}(t)$ has the Itô differential

$$\mathrm{d}\mathbf{u}(t) + \widetilde{F}_0(t)\mathrm{d}t + J(t)\mathrm{d}t = S(t)\mathrm{d}W(t) \tag{2.50}$$

in $\mathbb{L}^2(\Omega \times [0, T]; \mathbb{V}')$.

2. Local Minty-Browder technique.

Define

$$R(t) := \frac{16}{\nu^3}\int_0^t \|\mathbf{v}\|^4_{\mathbb{L}^4(G)}\mathrm{d}s + \left(M_1 + \sqrt{\lambda(\mathbb{H})M_2}\right)t, \tag{2.51}$$

where $\mathbf{v}(t, \omega, x)$ is an any adapted process in $\mathbb{L}^\infty(0, T; \mathbb{H}_m)$ with $m < n$.

By applying Itô's Lemma (Theorem 4.4.10 of [2] or Section 4 of [33]) to $e^{-R(t)}|\mathbf{u}_n(t)|^2$,

$$\begin{aligned}\mathrm{d}\left[e^{-R(t)}|\mathbf{u}_n(t)|^2\right] &= e^{-R(t)}\mathrm{d}|\mathbf{u}_n(t)|^2 - \dot{R}(t)e^{-R(t)}|\mathbf{u}_n(t)|^2 \\ &= e^{-R(t)}\Big[\big(-2\widetilde{F}(\mathbf{u}_n(t)), \mathbf{u}_n(t)\big)\mathrm{d}t - 2\int_{\mathbb{H}} (\phi_n(\mathbf{u}_n(s), x), \mathbf{u}_n(s))\lambda(\mathrm{d}x)\mathrm{d}t\end{aligned}$$

$$+ 2(\psi_n(t, \mathbf{u}_n(t))\mathrm{d}\mathrm{W}_n(t), \mathbf{u}_n(t)) + |\psi_n(t, \mathbf{u}_n(t))|^2_{\mathrm{L_Q}}\mathrm{d}t\Big]$$
$$- \dot{R}(t)e^{-R(t)}|\mathbf{u}_n(t)|^2\mathrm{d}t. \tag{2.52}$$

Integrating (2.52) from 0 to T,

$$\begin{aligned}
&e^{-R(T)}|\mathbf{u}_n(T)|^2 - |\mathbf{u}_n(0)|^2 \\
&= \int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}(\mathbf{u}_n(t)), \mathbf{u}_n(t)\big)\mathrm{d}t \\
&\quad - 2\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi_n(\mathbf{u}_n(t), x), \mathbf{u}_n(t))\lambda(\mathrm{d}x)\mathrm{d}t \\
&\quad + 2\int_0^T e^{-R(t)}(\psi_n(t, \mathbf{u}_n(t))\mathrm{d}\mathrm{W}_n(t), \mathbf{u}_n(t)) \\
&\quad + \int_0^T e^{-R(t)}|\psi_n(t, \mathbf{u}_n(t))|^2_{\mathrm{L_Q}}\mathrm{d}t - \int_0^T \dot{R}(t)e^{-R(t)}|\mathbf{u}_n(t)|^2\mathrm{d}t.
\end{aligned} \tag{2.53}$$

Taking expectation on both sides and using the fact that the third term is martingale,

$$\begin{aligned}
&\mathrm{E}\Big(e^{-R(T)}|\mathbf{u}_n(T)|^2\Big) - \mathrm{E}|\mathbf{u}_n(0)|^2 \\
&= \mathrm{E}\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}(\mathbf{u}_n(t)) - \dot{R}(t)\mathbf{u}_n(t), \mathbf{u}_n(t)\big)\mathrm{d}t \\
&\quad - 2\mathrm{E}\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi_n(\mathbf{u}_n(t), x), \mathbf{u}_n(t))\lambda(\mathrm{d}x)\mathrm{d}t \\
&\quad + \mathrm{E}\int_0^T e^{-R(t)}|\psi_n(t, \mathbf{u}_n(t))|^2_{\mathrm{L_Q}}\mathrm{d}t.
\end{aligned} \tag{2.54}$$

Taking liminf on both sides of (2.54) and by means of lower semi-continuity of $\mathbb{L}^2$-norm and strong convergence of the initial data $\mathbf{u}_n(0)$,

$$\begin{aligned}
&\liminf_{n\to\infty} \mathrm{E}\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}(\mathbf{u}_n(t)) - \dot{R}(t)\mathbf{u}_n(t), \mathbf{u}_n(t)\big)\mathrm{d}t \\
&\quad - \liminf_{n\to\infty} 2\mathrm{E}\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi_n(\mathbf{u}_n(t), x), \mathbf{u}_n(t))\lambda(\mathrm{d}x)\mathrm{d}t \\
&\quad + \liminf_{n\to\infty} \mathrm{E}\int_0^T e^{-R(t)}|\psi_n(t, \mathbf{u}_n(t))|^2_{\mathrm{L_Q}}\mathrm{d}t \\
&= \liminf_{n\to\infty}\Big[\mathrm{E}\Big(e^{-R(T)}|\mathbf{u}_n(T)|^2\Big) - \mathrm{E}|\mathbf{u}_n(0)|^2\Big] \\
&\geq \mathrm{E}\Big(e^{-R(T)}|\mathbf{u}(T)|^2\Big) - \mathrm{E}|\mathbf{u}(0)|^2 \\
&= \mathrm{E}\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}_0(t) - \dot{R}(t)\mathbf{u}(t), \mathbf{u}(t)\big)\mathrm{d}t. \\
&\quad - 2\mathrm{E}\int_0^T e^{-R(t)}(\mathrm{J}(t), \mathbf{u}(t))\mathrm{d}t + \mathrm{E}\int_0^T e^{-R(t)}|\mathrm{S}(t)|^2_{\mathrm{L_Q}}\mathrm{d}t
\end{aligned} \tag{2.55}$$

By Theorem 2.2 with $p = 4$, we have

$$
\begin{aligned}
&2\mathrm{E}\int_0^T e^{-R(t)}\big(\widetilde{\mathrm{F}}(\mathbf{u}_n(t)) - \widetilde{\mathrm{F}}(\mathbf{v}(t)), \mathbf{u}_n(t) - \mathbf{v}(t)\big)\mathrm{d}t \\
&\quad + 2\mathrm{E}\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi_n(\mathbf{u}_n(t), x) - \phi_n(\mathbf{v}(t), x), \mathbf{u}_n(t) - \mathbf{v}(t))\lambda(\mathrm{d}x)\mathrm{d}t \\
&\quad - \mathrm{E}\int_0^T e^{-R(t)}|\psi_n(t, \mathbf{u}_n(t)) - \psi_n(t, \mathbf{v}(t))|^2_{\mathrm{L_Q}}\mathrm{d}t \\
&\quad + \mathrm{E}\int_0^T e^{-R(t)}\dot{R}(t)|\mathbf{u}_n(t) - \mathbf{v}(t)|^2\mathrm{d}t \geq 0.
\end{aligned} \tag{2.56}
$$

Now, rearrange the terms of (2.56) to get

$$
\begin{aligned}
&\mathrm{E}\Big[\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}(\mathbf{u}_n(t)) - \dot{R}(t)\mathbf{u}_n(t), \mathbf{u}_n(t)\big)\mathrm{d}t \\
&\quad - 2\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi_n(\mathbf{u}_n(t), x), \mathbf{u}_n(t))\lambda(\mathrm{d}x)\mathrm{d}t \\
&\quad + \int_0^T e^{-R(t)}|\psi_n(t, \mathbf{u}_n(t))|^2_{\mathrm{L_Q}}\mathrm{d}t\Big] \\
&\leq \mathrm{E}\Big[\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}(\mathbf{u}_n(t)) - \dot{R}(t)\mathbf{u}_n(t), \mathbf{v}(t)\big)\mathrm{d}t \\
&\quad + \int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}(\mathbf{v}(t)) - \dot{R}(t)\mathbf{v}(t), \mathbf{u}_n(t) - \mathbf{v}(t)\big)\mathrm{d}t \\
&\quad - 2\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi_n(\mathbf{u}_n(t), x), \mathbf{v}(t))\lambda(\mathrm{d}x)\mathrm{d}t \\
&\quad - 2\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi_n(\mathbf{v}(t), x), \mathbf{u}_n(t) - \mathbf{v}(t))\lambda(\mathrm{d}x)\mathrm{d}t \\
&\quad + \int_0^T e^{-R(t)}\big(2\psi_n(t, \mathbf{u}_n(t)) - \psi_n(t, \mathbf{v}(t)), \psi_n(t, \mathbf{v}(t))\big)\mathrm{d}t\Big].
\end{aligned} \tag{2.57}
$$

Taking liminf on both sides and applying (2.55)

$$
\begin{aligned}
&\mathrm{E}\Big[\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}_0(t) - \dot{R}(t)\mathbf{u}(t), \mathbf{u}(t)\big)\mathrm{d}t \\
&\quad - 2\int_0^T e^{-R(t)}(\mathrm{J}(t), \mathbf{u}(t))\mathrm{d}t + \int_0^T e^{-R(t)}|\mathrm{S}(t)|^2_{\mathrm{L_Q}}\mathrm{d}t\Big] \\
&\leq \mathrm{E}\Big[\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}_0(t) - \dot{R}(t)\mathbf{u}(t), \mathbf{v}(t)\big)\mathrm{d}t \\
&\quad - 2\int_0^T e^{-R(t)}(\mathrm{J}(t), \mathbf{v}(t))\mathrm{d}t \\
&\quad - 2\int_0^T e^{-R(t)}\int_{\mathbb{H}}(\phi(\mathbf{v}(t), x), \mathbf{u}(t) - \mathbf{v}(t))\lambda(\mathrm{d}x)\mathrm{d}t
\end{aligned}
$$

$$+\int_0^T e^{-R(t)}\big(-2\widetilde{\mathrm{F}}(\mathbf{v}(t)) - \dot{R}(t)\mathbf{v}(t), \mathbf{u}(t)-\mathbf{v}(t)\big)\mathrm{d}t$$
$$+2\int_0^T e^{-R(t)}\big(\mathrm{S}(t), \psi(t,\mathbf{v}(t))\big)\mathrm{d}t - \int_0^T e^{-R(t)}|\psi(t,\mathbf{v}(t)|^2_{\mathrm{L_Q}}\mathrm{d}t\Bigg]. \tag{2.58}$$

By reorganizing terms, one can deduce

$$\mathrm{E}\Bigg[\int_0^T e^{-r(t)}\big(-2\big(\widetilde{\mathrm{F}}_0(t) - \widetilde{\mathrm{F}}(\mathbf{v}(t))\big) - \dot{R}(t)(\mathbf{u}(t)-\mathbf{v}(t)), \mathbf{u}(t)-\mathbf{v}(t)\big)\mathrm{d}t$$
$$-2\int_0^T e^{-R(t)}(\mathrm{J}(t) - \int_{\mathbb{H}}\phi(\mathbf{v}(t),x)\lambda(\mathrm{d}x), \mathbf{u}(t)-\mathbf{v}(t))\mathrm{d}t$$
$$+\int_0^T e^{-R(t)}|\mathrm{S}(t) - \psi(t,\mathbf{v}(t))|^2_{\mathrm{L_Q}}\mathrm{d}t\Bigg] \le 0. \tag{2.59}$$

(2.59) holds for any $\mathbf{v}\in\mathbb{L}^2(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}_m))$ and any $m\in\mathbb{N}$. By using density argument (see [26], [9]) we can verify that above inequality remains true for any $\mathbf{v}\in\mathbb{L}^2(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^2(0,T;\mathbb{V}))$.

It is clear that for $\mathbf{u}(t)=\mathbf{v}(t)$ we have $\mathrm{S}(t)=\psi(t,\mathbf{v}(t))$. Let $\mathbf{v}=\mathbf{u}+\delta\mathbf{w}$, where $\delta>0$ and $\mathbf{w}\in\mathbb{L}^2(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^2(0,T;\mathbb{V}))$. By substituting this $\mathbf{v}$ into (2.59),

$$\mathrm{E}\int_0^T e^{-R(t)}\Bigg[\big(-2\big(\mathrm{F}_0(t) - \mathrm{F}(\mathbf{u}(t)+\delta\mathbf{w}(t))\big) - \dot{R}(t)\delta\mathbf{w}(t), \delta\mathbf{w}(t)\big)$$
$$-2(\mathrm{J}(t) - \int_{\mathbb{H}}\phi(\mathbf{u}(t)+\delta\mathbf{w}(t),x)\lambda(\mathrm{d}x), \delta\mathbf{w}(t))\Bigg]\mathrm{d}t \le 0. \tag{2.60}$$

Definition of $\mathrm{F}(\mathbf{u}) := \nu\mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u})$ and the first term appearing in left-hand side of (2.60) give

$$\mathrm{E}\int_0^T e^{-R(t)}\big(-2\mathrm{F}_0(t) + 2\nu\mathbf{A}(\mathbf{u}(t)+\delta\mathbf{w}(t)) + 2\mathbf{B}(\mathbf{u}(t)+\delta\mathbf{w}(t)), \delta\mathbf{w}(t)\big)\mathrm{d}t$$
$$-\delta^2\mathrm{E}\int_0^T e^{-R(t)}\dot{R}(t)|\mathbf{w}(t)|^2\mathrm{d}t$$
$$=\mathrm{E}\int_0^T e^{-R(t)}\Bigg[\big(-2\mathrm{F}_0(t) + 2\nu\mathbf{A}\mathbf{u}(t) + 2\nu\delta\mathbf{A}\mathbf{w}(t) + 2\mathbf{B}(\mathbf{u}(t)), \delta\mathbf{w}(t)\big)$$
$$+\big(\delta(\mathbf{B}(\mathbf{u}(t),\mathbf{w}(t)) + \mathbf{B}(\mathbf{w}(t),\mathbf{u}(t))) + \delta^2\mathbf{B}(\mathbf{u}(t)), \delta\mathbf{w}(t)\big)\Bigg]\mathrm{d}t$$
$$-\delta^2\mathrm{E}\int_0^T e^{-R(t)}\dot{R}(t)|\mathbf{w}(t)|^2\mathrm{d}t. \tag{2.61}$$

Consider

$$\lim_{\delta\to 0}\int_{\mathbb{H}}(\phi(\mathbf{u}(t)+\delta\mathbf{w}(t),x) - \phi(\mathbf{u}(t),x), \mathbf{w}(t))\lambda(\mathrm{d}x)$$
$$\le |\mathbf{w}(t)|\sqrt{\lambda(\mathbb{H})}\left[\lim_{\delta\to 0}\left(\int_{\mathbb{H}}|\phi(\mathbf{u}(t)+\delta\mathbf{w}(t),x) - \phi(\mathbf{u}(t),x)|^2\lambda(\mathrm{d}x)\right)^{\frac{1}{2}}\right]$$
$$\le \lim_{\delta\to 0}\delta\mathrm{M}_2|\mathbf{w}(t)|^2 = 0. \tag{2.62}$$

(2.62) holds due to Cauchy-Schwartz inequality and condition B2.

Apply (2.61) and (2.62) for inequality (2.60), divide by δ and letting $\delta \to 0$,

$$\mathrm{E}\int_0^T e^{-R(t)}\Big(- \mathrm{F}_0(t) - 2\mathrm{J}(t) + \nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t)) + 2\int_{\mathbb{H}} \phi(\mathbf{u}(t), x)\lambda(\mathrm{d}x), \mathbf{w}(t)\Big)\mathrm{d}t \le 0. \tag{2.63}$$

Since this inequality holds for any $\mathbf{w}(t) \in \mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H}) \cap \mathbb{L}^2(0, T; \mathbb{V}))$, we have $\mathrm{F}_0(t) + 2\mathrm{J}(t) = \mathrm{F}(\mathbf{u}(t)) + 2\int_{\mathbb{H}} \phi(\mathbf{u}(t), x)\lambda(\mathrm{d}x)$.

Therefore existence of the strong pathwise solution of (2.48) has been proved.

Now let us prove the uniqueness of the solution of (2.48).

Suppose $\mathbf{v}(t) \in \mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H}) \cap \mathbb{C}(0, T; \mathbb{H})) \cap \mathbb{L}^2(\Omega \times [0, T]; \mathbb{V})$ be another solution of (2.48) on the interval $[0, \mu_1(\omega))$. Then $\mathbf{w}(t) = \mathbf{u}(t) - \mathbf{v}(t)$ satisfies the stochastic differential equation on $[0, \mu_1(\omega))$ in $\mathbb{L}^2(\Omega; \mathbb{L}^2(0, T; \mathbb{V}'))$:

$$\mathrm{d}\mathbf{w}(t) = -(\mathrm{F}(\mathbf{u}(t)) - \mathrm{F}(\mathbf{v}(t)))\mathrm{d}t - \int_{\mathbb{H}} \big(\phi(\mathbf{u}(t), x) - \phi(\mathbf{v}(t), x)\big)\lambda(\mathrm{d}x)\mathrm{d}t + (\psi(t, \mathbf{u}(t)) - \psi(t, \mathbf{v}(t)))\mathrm{d}\mathrm{W}(t) \tag{2.64}$$

By applying Itô's Lemma (Theorem 4.4.10 of [2] or Section 4 of [33]) to $e^{-R(t)}|\mathbf{w}(t)|^2$,

$$\begin{aligned}\mathrm{d}\big(e^{-R(t)}|\mathbf{w}(t)|^2\big) = &-e^{-R(t)}\dot{R}(t)|\mathbf{w}(t)|^2\mathrm{d}t - 2e^{-R(t)}\big(\mathrm{F}(\mathbf{u}(t)) - \mathrm{F}(\mathbf{v}(t)), \mathbf{w}(t)\big)\mathrm{d}t \\ &- 2e^{-R(t)}\int_{\mathbb{H}} \big(\phi_{\mathrm{d}}, \mathbf{w}(t)\big)\lambda(\mathrm{d}x)\mathrm{d}t + 2e^{-R(t)}(\psi_{\mathrm{d}}\mathrm{d}\mathrm{W}(t), \mathbf{w}(t)) \\ &+ e^{-R(t)}\mathrm{Tr}(\psi_{\mathrm{d}}\mathrm{Q}\psi_{\mathrm{d}}^*)\mathrm{d}t,\end{aligned} \tag{2.65}$$

where $\psi_{\mathrm{d}} = \psi(t, \mathbf{u}(t)) - \psi(t, \mathbf{v}(t))$ and $\phi_{\mathrm{d}} = \phi(\mathbf{u}(t), x) - \phi(\mathbf{v}(t), x)$. Integrating up to $t \le T$, taking the expectation, and using the fact that third term is a martingale in (2.65),

$$\begin{aligned}\mathrm{E}\big(e^{-R(t)}|\mathbf{w}(t)|^2\big) \le \mathrm{E}|\mathbf{w}(0)|^2 &- 2\mathrm{E}\int_0^t e^{-R(s)}\dot{R}(s)|\mathbf{w}(s)|^2\mathrm{d}s \\ &- 2\mathrm{E}\int_0^t e^{-R(s)}\big(\mathrm{F}(\mathbf{u}(s)) - \mathrm{F}(\mathbf{v}(s)), \mathbf{w}(s)\big)\mathrm{d}s \\ &- 2\mathrm{E}\int_0^t e^{-R(s)}\int_{\mathbb{H}} \big(\phi_{\mathrm{d}}, \mathbf{w}(s)\big)\lambda(\mathrm{d}x)\mathrm{d}s \\ &+ \mathrm{E}\int_0^t e^{-R(s)}|\psi_{\mathrm{d}}|^2_{\mathrm{L_Q}}\mathrm{d}s\end{aligned} \tag{2.66}$$

By local monotonicity argument (Theorem 2.2) with ($p = 4$), we have

$$\mathrm{E}\big(e^{-R(t)}|\mathbf{w}(t)|^2\big) \le \mathrm{E}|\mathbf{w}(0)|^2 \quad \text{for } 0 \le t < \mu_1. \tag{2.67}$$

(2.67) immediately gives us the uniqueness of the solution of (2.48). □

Theorem 2.5. *Let* $\mathbf{f} \in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{V}'))$ *and initial value* $\mathbf{u}_0$ *satisfies* $\mathrm{E}|\mathbf{u}_0|^4 < \infty$. *Suppose that* $\lambda(\mathbb{H}) < \infty$ *and the conditions associated with* $\psi(.,.)$ *and* $\phi(.,.)$ (*i.e.,* *A*1-*A*2 *and B*1-*B*3) *hold. Then the model* (2.14) *has a unique adapted cádlág process* $\mathbf{u}(t, \omega, x)$ *on* $[0, T]$ *belonging to* $\mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H}) \cap \mathbb{L}^2(0, T; \mathbb{V}))$ *for any fixed* $T > 0$.

Proof. Let $\{\mu_i, i = 1, \ldots, n\}$ be the jump times associated with compound Poisson process $\{P(t), t > 0\}$, where $P(t) = \int_{\mathbb{H}} xN(ds, dx)$.

From Theorem 2.4, there exists a unique strong adapted process $\mathbf{u}(t)$ in $\mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H}) \cap \mathbb{C}(0, T; \mathbb{H})) \cap \mathbb{L}^2(\Omega \times [0, T]; \mathbb{V})$ on the interval $[0, \mu_1(\omega))$. Now we recursively construct the solution $\mathbf{u}(t)$ of (2.14) as follows.

Define on $[0, \mu_1]$

$$\mathbf{u}^{[1]}(t) = \begin{cases} \mathbf{u}(t) & \text{for } t < \mu_1 \\ \mathbf{u}(\mu_1^-) + \phi(\mathbf{u}(\mu_1^-), \Delta P(\mu_1)) & \text{for } t = \mu_1 \end{cases} \tag{2.68}$$

Now suppose that $P\{\omega \in \Omega : \mu_1 < \infty\} = 1$. Define $\acute{\mathbf{u}}(0) = \mathbf{u}^{[1]}(\mu_1)$, and $\mathcal{F}_t' = \mathcal{F}_{\mu_1+t}$. Let $\acute{P}(t)$ be the compound Poisson process which starts from time μ_1.

Since we don't have jumps during the time interval (μ_1, μ_2), the stochastic differential equation (2.48) has an unique strong pathwise solution $\acute{\mathbf{u}}(t - \mu_1) \in \mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H}) \cap \mathbb{C}(0, T; \mathbb{H})) \cap \mathbb{L}^2(\Omega \times [0, T]; \mathbb{V})$ with initial value $\acute{\mathbf{u}}(0)$ on $[0, \mu_2 - \mu_1)$. Then,

$$\mathbf{u}^{[2]}(t) = \begin{cases} \mathbf{u}^{[1]}(t) & \text{for } t \le \mu_1 \\ \acute{\mathbf{u}}(t - \mu_1) & \text{for } \mu_1 \le t \le \mu_2 \\ \acute{\mathbf{u}}((\mu_2 - \mu_1)^-) + \phi(\acute{\mathbf{u}}((\mu_2 - \mu_1)^-), \Delta \acute{P}(\mu_2)) & \text{for } t = \mu_2 \end{cases} \tag{2.69}$$

Since we have a finite number of jumps on $[0.T]$, by repeating the above process n times, we can obtain $\mathbf{u}^{[n]}(t)$.

$\mathbf{u}^{[n]}(t)$ is clearly a adapted cádlág process which solves (2.14). Uniqueness of $\mathbf{u}^{[n]}(t)$ follows from the second part of the Theorem 2.4 and the interlacing structure of the solution. □

2.3.2. Lévy Measure $\lambda(.)$ *is* σ*-Finite.* In this subsection, we use the approach method in [37] to prove existence and uniqueness of strong pathwise solution of (2.14) with σ-finite Lévy measure. This method of deriving strong convergence is also used in [49].

Note 2.4. In [49] solvability of the stochastic Navier-Stokes equation with additive Wiener noise and multiplicative Lévy noise for the case of σ-finite Lévy measure was discussed in the domain $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We extend these results by adding multiplicative Wiener noise and Lévy noise to stochastic Navier-Stokes equation in bounded or unbounded domain $G \subset \mathbb{R}^2$.

Theorem 2.6. *Let* $\mathbf{f} \in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{V}'))$ *and initial value* $\mathbf{u}_0$ *satisfies* $E|\mathbf{u}_0|^4 < \infty$. *Suppose that* $\lambda(.)$ *is* σ*-finite and the conditions associated with* $\psi(., .)$ *and* $\phi(., .)$ (*i.e., A*1-*A*2 *and B*1-*B*3) *hold. Then the model* (2.14) *has a unique adapted cádlág strong pathwise solution* $\mathbf{u}(t, \omega, x)$ *on* $[0, T]$ *in* $\mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{H}) \cap \mathbb{L}^2(0, T; \mathbb{V}))$ *for any fixed* $T > 0$.

Proof. Now consider the following stochastic Navier-Stokes equation with Itô-Lévy noise

$$\begin{aligned}&\mathrm{d}\mathbf{u}_n(t)+\mathbf{A}\mathbf{u}_n(t)\mathrm{d}t+\mathbf{B}(\mathbf{u}_n(t))\mathrm{d}t\\&\quad=\mathbf{f}(t)\mathrm{d}t+\psi(t,\mathbf{u}_n(t))\mathrm{d}\mathrm{W}(t)+\int_{\mathbb{Z}_n}\phi(\mathbf{u}_n(t^-),x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x),\end{aligned}\tag{2.70}$$

with $\mathbf{u}_n(0)=\mathbf{u}_0$ and for all $n\geq 1$.

Theorem 2.5 implies that Equation (2.70) has a unique adapted cádlág process $\mathbf{u}_n(t)$ belonging in $\mathbb{L}^2(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H}))\cap\mathbb{L}^2(0,T;\mathbb{V})$ for any $n\geq 1$.

We define $\tau_N^n:=\inf\{t:|\mathbf{u}_n(t)|^2\vee\int_0^t\|\mathbf{u}_n(s)\|^2\mathrm{d}s\geq N\}$, for any fixed $N\geq 1$ and $n\geq 1$.

Consider the following stochastic differential equation with $n\geq m$:

$$\begin{aligned}&\mathrm{d}(\mathbf{u}_n(t)-\mathbf{u}_m(t))+\mathbf{A}(\mathbf{u}_n(t)-\mathbf{u}_m(t))\mathrm{d}t+[\mathbf{B}(\mathbf{u}_n(t))-\mathbf{B}(\mathbf{u}_m(t))]\,\mathrm{d}t\\&\quad=[\psi(t,\mathbf{u}_n(t))-\psi(t,\mathbf{u}_m(t))]\,\mathrm{d}\mathrm{W}(t)\\&\qquad+\left[\int_{\mathbb{Z}_n}\phi(\mathbf{u}_n(t^-),x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x)-\int_{\mathbb{Z}_m}\phi(\mathbf{u}_m(t^-),x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x)\right].\end{aligned}\tag{2.71}$$

Applying Itô's Lemma (Theorem 4.4.10 of [2] or Section 4 of [33]) to $|\mathbf{u}_n(t)-\mathbf{u}_m(t)|^2$,

$$\begin{aligned}&|\mathbf{u}_n(t\wedge\tau_N^n)-\mathbf{u}_m(t\wedge\tau_N^n)|^2+2\nu\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\\&\leq\int_0^{t\wedge\tau_N^n}|\,(\mathbf{B}(\mathbf{u}_n(s))-\mathbf{B}(\mathbf{u}_m(s)),\mathbf{u}_n(s)-\mathbf{u}_m(s))\,|\mathrm{d}s\\&\quad+2\int_0^{t\wedge\tau_N^n}([\psi(s,\mathbf{u}_n(s))-\psi(s,\mathbf{u}_m(s))]\,\mathrm{d}\mathrm{W}(s),\mathbf{u}_n(s)-\mathbf{u}_m(s))\\&\quad+\int_0^{t\wedge\tau_N^n}|\psi(s,\mathbf{u}_n(s))-\psi(s,\mathbf{u}_m(s))|^2_{\mathrm{L_Q}}\mathrm{d}s\\&\quad+2\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_m}(\phi(\mathbf{u}_n(s^-),x)-\phi(\mathbf{u}_m(s^-),x),\mathbf{u}_n(s)-\mathbf{u}_m(s))\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x)\\&\quad+2\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_n\backslash\mathbb{Z}_m}(\phi(\mathbf{u}_n(s^-),x),\mathbf{u}_n(s)-\mathbf{u}_m(s))\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x)\\&\quad+\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_m}|\phi(\mathbf{u}_n(s^-),x)-\phi(\mathbf{u}_m(s^-),x)|^2N(\mathrm{d}s,\mathrm{d}x)\\&\quad+\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_m^c}|\phi(\mathbf{u}_n(s^-),x)|^2N(\mathrm{d}s,\mathrm{d}x)\end{aligned}\tag{2.72}$$

Applying (2.11) to the first term of the right-hand side of (2.72) with Young's inequality,

$$\begin{aligned}&|\mathbf{u}_n(t\wedge\tau_N^n)-\mathbf{u}_m(t\wedge\tau_N^n)|^2+\nu\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\\&\leq\frac{K_B^2}{4\nu}\int_0^{t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2\|\mathbf{u}_n(s)\|^2\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^{t\wedge\tau_N^n}([\psi(s,\mathbf{u}_n(s))-\psi(s,\mathbf{u}_m(s))]\,\mathrm{dW}(s),\mathbf{u}_n(s)-\mathbf{u}_m(s))\\
&+\int_0^{t\wedge\tau_N^n}|\psi(s,\mathbf{u}_n(s))-\psi(s,\mathbf{u}_m(s))|_{\mathrm{L_Q}}^2\mathrm{d}s\\
&+2\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_m}(\phi(\mathbf{u}_n(s^-),x)-\phi(\mathbf{u}_m(s^-),x),\mathbf{u}_n(s)-\mathbf{u}_m(s))\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x)\\
&+2\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_n\setminus\mathbb{Z}_m}(\phi(\mathbf{u}_n(s^-),x),\mathbf{u}_n(s)-\mathbf{u}_m(s))\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x)\\
&+\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_m}|\phi(\mathbf{u}_n(s^-),x)-\phi(\mathbf{u}_m(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s\\
&+\int_0^{t\wedge\tau_N^n}\int_{\mathbb{Z}_m^c}|\phi(\mathbf{u}_n(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s,
\end{aligned}
\tag{2.73}
$$

Let $\Upsilon(t\wedge\tau_N^n)$ is the summation of last six terms of (2.73). Now applying Gronwall's inequality to (2.73),

$$
\begin{aligned}
&|\mathbf{u}_n(t\wedge\tau_N^n)-\mathbf{u}_m(t\wedge\tau_N^n)|^2+\nu\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\\
&\quad\le|\Upsilon(t\wedge\tau_N^n)|\exp\left(\frac{K_B^2}{4\nu}\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)\|^2\mathrm{d}s\right)\\
&\quad\le|\Upsilon(t\wedge\tau_N^n)|\exp\left(\frac{K_B^2}{4\nu}Nt\right).
\end{aligned}
\tag{2.74}
$$

Then apply Burkholder-Gundy-Davis inequality, Cauchy-Schwartz inequality, and Young's inequality with conditions A2, B2 to $\mathrm{E}[\sup_{s\le t}|\Upsilon(s\wedge\tau_N^n)|]$:

$$
\begin{aligned}
\mathrm{E}\left[\sup_{s\le t}|\Upsilon(s\wedge\tau_N^n)|\right]&\le 2\sqrt{2}\mathrm{E}\left(\int_0^t\left|\psi_{s\wedge\tau_N^n}^{nm}\right|_{\mathrm{L_Q}}^2\left|\mathbf{u}_{s\wedge\tau_N^n}^{nm}\right|^2\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\mathrm{E}\int_0^t\left|\psi_{s\wedge\tau_N^n}^{nm}\right|_{\mathrm{L_Q}}^2\mathrm{d}s+2\sqrt{2}\mathrm{E}\left(\int_0^t\int_{\mathbb{Z}_m}\left|\phi_{s\wedge\tau_N^n}^{nm}\right|^2\left|\mathbf{u}_{s\wedge\tau_N^n}^{nm}\right|^2\lambda(\mathrm{d}x)\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+2\sqrt{2}\mathrm{E}\left(\int_0^t\int_{\mathbb{Z}_n\setminus\mathbb{Z}_m}|\phi(\mathbf{u}_n(s\wedge\tau_N^n-),x)|^2\left|\mathbf{u}_{s\wedge\tau_N^n}^{nm}\right|^2\lambda(\mathrm{d}x)\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\mathrm{E}\int_0^t\int_{\mathbb{Z}_m}\left|\phi_{s\wedge\tau_N^n}^{nm}\right|^2\lambda(\mathrm{d}x)\mathrm{d}s\\
&\quad+\mathrm{E}\int_0^t\int_{\mathbb{Z}_m^c}|\phi(\mathbf{u}_n(s\wedge\tau_N^n-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s\\
&\le\frac{3}{\mathrm{C}_5}\mathrm{E}\left(\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_s^{nm}|^2\right)+(\mathrm{C}_5+1)\mathrm{E}\int_0^t\left|\psi_{s\wedge\tau_N^n}^{nm}\right|_{\mathrm{L_Q}}^2\mathrm{d}s\\
&\quad+(\mathrm{C}_5+1)\mathrm{E}\int_0^t\int_{\mathbb{Z}_m}\left|\phi_{s\wedge\tau_N^n}^{nm}\right|^2\lambda(\mathrm{d}x)\mathrm{d}s\\
&\quad+\mathrm{C}_5\mathrm{E}\int_0^t\int_{\mathbb{Z}_n\setminus\mathbb{Z}_m}|\phi(\mathbf{u}_n(s\wedge\tau_N^n-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathrm{E}\int_0^t \int_{\mathbb{Z}_m^c} |\phi(\mathbf{u}_n(s\wedge\tau_N^n-), x)|^2\,\lambda(\mathrm{d}x)\mathrm{d}s\\
&\le \frac{3}{\mathrm{C}_5}\mathrm{E}\left(\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_s^{nm}|^2\right) + (\mathrm{C}_5+1)\mathrm{M}_1\mathrm{E}\int_0^t |\mathbf{u}_{s\wedge\tau_N^n}^{nm}|^2\mathrm{d}s\\
&+ (\mathrm{C}_5+1)\mathrm{M}_2\mathrm{E}\int_0^t \left|\mathbf{u}_{s\wedge\tau_N^n}^{nm}\right|^2\mathrm{d}s\\
&+ (\mathrm{C}_5+1)\mathrm{E}\int_0^t \int_{\mathbb{Z}_m^c} |\phi(\mathbf{u}_n(s\wedge\tau_N^n-), x)|^2\,\lambda(\mathrm{d}x)\mathrm{d}s,
\end{aligned}
\tag{2.75}
$$

where $\psi_t^{nm} = \psi(t,\mathbf{u}_n(t)) - \psi(t,\mathbf{u}_m(t))$, $\phi_t^{nm} = \phi(\mathbf{u}_n(t),x) - \phi(\mathbf{u}_m(t),x)$, $\mathbf{u}_t^{nm} = \mathbf{u}_n(t) - \mathbf{u}_m(t)$ and $\mathrm{C}_5 = \frac{\sqrt{2}}{\epsilon}\exp(\frac{K_B^2}{4\nu}Nt)$. Then,

$$
\begin{aligned}
&\mathrm{E}\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2 + \nu\int_0^{t\wedge\tau_N^n}\mathrm{E}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\\
&\le 3\sqrt{2}\epsilon\mathrm{E}\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2\\
&+ \mathrm{C}_6\int_0^t \mathrm{E}\sup_{\acute{s}\le s\wedge\tau_N^n}|\mathbf{u}_n(\acute{s})-\mathbf{u}_m(\acute{s})|^2\mathrm{d}s\\
&+ \mathrm{C}_7\mathrm{E}\int_0^t\int_{\mathbb{Z}_m^c}|\phi(\mathbf{u}_n(s\wedge\tau_N^n-),x)|^2\,\lambda(\mathrm{d}x)\mathrm{d}s.
\end{aligned}
\tag{2.76}
$$

and

$$
\begin{aligned}
&\mathrm{E}\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2 + \nu\int_0^{t\wedge\tau_N^n}\mathrm{E}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\\
&\le \frac{\mathrm{C}_6}{1-3\sqrt{2}\epsilon}\int_0^t \mathrm{E}\sup_{\acute{s}\le s\wedge\tau_N^n}|\mathbf{u}_n(\acute{s})-\mathbf{u}_m(\acute{s})|^2\mathrm{d}s\\
&+ \frac{\mathrm{C}_7}{1-3\sqrt{2}\epsilon}\mathrm{E}\int_0^t\int_{\mathbb{Z}_m^c}|\phi(\mathbf{u}_n(s\wedge\tau_N^n-),x)|^2\,\lambda(\mathrm{d}x)\mathrm{d}s.
\end{aligned}
\tag{2.77}
$$

Now apply Gronwall's lemma and condition B2 to (2.78):

$$
\begin{aligned}
&\mathrm{E}\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2 + \nu\int_0^{t\wedge\tau_N^n}\mathrm{E}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\\
&\le \exp\left\{\frac{\mathrm{C}_6+\mathrm{C}_7}{1-3\sqrt{2}\epsilon}t\right\}\\
&\times \mathrm{E}\int_0^t\int_{\mathbb{Z}_m^c}|\phi(\mathbf{u}_n(s\wedge\tau_N^n-),x)|^2\,\lambda(\mathrm{d}x)\mathrm{d}s\\
&\le \exp\left\{\frac{\mathrm{C}_6+\mathrm{C}_7}{1-3\sqrt{2}\epsilon}t\right\}\sup_{|\mathbf{u}|\le\sqrt{N}}\int_{\mathbb{Z}_m^c}|\phi(\mathbf{u},x)|^2\,\lambda(\mathrm{d}x).
\end{aligned}
\tag{2.78}
$$

Then we have

$$\lim_{m\to\infty}\left(\mathrm{E}\left[\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2\right]+\nu\int_0^{t\wedge\tau_N^n}\mathrm{E}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\right)=0.$$

From Theorem 2.3,

$$\int_\Omega\left(\sup_{0\le s\le t}|\mathbf{u}_n(s)|^2\vee\int_0^t\|\mathbf{u}_n(s)\|^2\mathrm{d}s\right)\mathrm{P}(\mathrm{d}\omega)\le C_T.$$

This implies

$$\mathrm{P}(t\ge\tau_N^n)=\mathrm{P}\left(\sup_{0\le s\le t}|\mathbf{u}_n(s)|^2\vee\int_0^t\|\mathbf{u}_n(s)\|^2\mathrm{d}s\ge N\right)\le\frac{C_T}{N}. \tag{2.79}$$

Consider the following two estimates:

$$\begin{aligned}
&\mathrm{E}\left[\sup_{s\le t}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|\right]\\
&\quad=\mathrm{E}\left[\sup_{s\le t}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|\chi_{\{t\le\tau_N^n\}}\right]\\
&\qquad+\mathrm{E}\left[\sup_{s\le t}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|\chi_{\{t>\tau_N^n\}}\right]\\
&\quad\le\mathrm{E}\left[\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|\right]+\left(\mathrm{E}\sup_{s\le t}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2\right)^{\frac{1}{2}}[\mathrm{P}(t>\tau_N^n)]^{\frac{1}{2}}\\
&\quad\le\left(\mathrm{E}\left[\sup_{s\le t\wedge\tau_N^n}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|^2\right]\right)^{\frac{1}{2}}+\frac{\sqrt{2}C_T}{\sqrt{N}}.
\end{aligned}\tag{2.80}$$

Last two inequalities of (2.80) is due to Cauchy-Schwartz inequality and (2.79). Similarly, we can get

$$\begin{aligned}
&\mathrm{E}\int_0^t\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|\mathrm{d}s\\
&\quad=\mathrm{E}\int_0^t\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|\mathrm{d}s\chi_{\{t\le\tau_N^n\}}\\
&\qquad+\mathrm{E}\int_0^t\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|\mathrm{d}s\chi_{\{t>\tau_N^n\}}\le\mathrm{E}\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|\mathrm{d}s\\
&\qquad+\left(t\mathrm{E}\int_0^t\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\right)^{\frac{1}{2}}[\mathrm{P}(t>\tau_N^n)]^{\frac{1}{2}}\\
&\quad\le\left(t\mathrm{E}\int_0^{t\wedge\tau_N^n}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|^2\mathrm{d}s\right)^{\frac{1}{2}}+\frac{\sqrt{2t}C_T}{\sqrt{N}}.
\end{aligned}\tag{2.81}$$

The first term of (2.80) and (2.81) approaches zero as $m \to \infty$ for fixed N. As $N \to \infty$, the second term of (2.80) and (2.81) approaches zero. Then,

$$\lim_{m\to\infty}\left(\mathrm{E}\left(\sup_{s\le t}|\mathbf{u}_n(s)-\mathbf{u}_m(s)|\right)+\nu\int_0^t \mathrm{E}\|\mathbf{u}_n(s)-\mathbf{u}_m(s)\|\mathrm{d}s\right)=0. \tag{2.82}$$

(2.82) shows that $\{\mathbf{u}_n(t)\}$ is a Cauchy sequence in the space $\mathbb{L}^1(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^1(0,T;\mathbb{V}))$. Accordingly, there exists a adapted cádlág process $\mathbf{u}(t)\in\mathbb{L}^1(\Omega;\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^1(0,T;\mathbb{V}))$ such that

$$\lim_{n\to\infty}\left(\mathrm{E}\left(\sup_{s\le T}|\mathbf{u}_n(s)-\mathbf{u}(s)|\right)+\nu\int_0^T \mathrm{E}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\right)=0.$$

Now let us prove that $\mathbf{u}(t)$ is the strong pathwise solution of (2.14). Let $\mathbf{v}\in \mathrm{D}(\mathbf{A})$. Then we have,

$$\mathrm{E}\sup_{t\le T}\left|\int_0^t(\mathbf{u}_n(s),\mathbf{A}\mathbf{v})\mathrm{d}s-\int_0^t(\mathbf{u}(s),\mathbf{A}\mathbf{v})\mathrm{d}s\right|\le T|\mathbf{A}\mathbf{v}|\mathrm{E}\left(\sup_{t\le T}|\mathbf{u}_n(t)-\mathbf{u}(t)|\right). \tag{2.83}$$

Define $\theta_N=\inf\{t:|\mathbf{u}(t)|\ge N\}$. Since (2.82) holds, we can extract a subsequence $\mathbf{u}_n(t)$ with relabeling such that $\lim_{n\to\infty}\sup_{s\le t}|\mathbf{u}_n(s)-\mathbf{u}(s)|=0$ a.s.

There exist $\tilde{n}(\omega,N)$ such that $\sup_{s\le t}|\mathbf{u}_n(s)-\mathbf{u}(s)|\le N$ for all $n\ge\tilde{n}(\omega,N)$.

Then we have $\sup_{s\le t\wedge\theta_N}|\mathbf{u}(s)|^{\frac12}|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}\le N$ for all $n\ge\tilde{n}(\omega,N)$ since $|\mathbf{u}(t\wedge\theta_N)|\le N$. Now let

$$\mathrm{S}_{\tilde{n}}(N)=\Bigg\{\omega:\sup_{s\le t\wedge\theta_N}|\mathbf{u}(s)|^{\frac12}|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}\le N \text{ and}$$
$$\sup_{s\le t\wedge\theta_N}|\mathbf{u}_n(s)-\mathbf{u}(s)|\le N \text{ for } n\ge\tilde{n}\Bigg\}.$$

$\{\mathrm{S}_{\tilde{n}}(N)\}_{\tilde{n}=1}^{\infty}$ is an increasing sequence of sets and $\cup_{\tilde{n}=1}^{\infty}\mathrm{S}_{\tilde{n}}(N)=\Omega$. Now,

$$\begin{aligned}
&\mathrm{E}\left[\sup_{t\le T}\left|\int_0^{t\wedge\theta_N}(\mathbf{B}(\mathbf{u}_n(s)),\mathbf{v})\mathrm{d}s-\int_0^{t\wedge\theta_N}(\mathbf{B}(\mathbf{u}(s)),\mathbf{v})\mathrm{d}s\right|\chi_{\mathrm{S}_{\tilde{n}}(N)}\right]\\
&\le \mathrm{E}\left[\int_0^{T\wedge\theta_N}|(\mathbf{B}(\mathbf{u}_n(s)-\mathbf{u}(s),\mathbf{u}_n(s)),\mathbf{v})|\,\mathrm{d}s\chi_{\mathrm{S}_{\tilde{n}}(N)}\right]\\
&\quad+\mathrm{E}\left[\int_0^{T\wedge\theta_N}|(\mathbf{B}(\mathbf{u}(s),\mathbf{u}_n(s)-\mathbf{u}(s)),\mathbf{v})|\,\mathrm{d}s\chi_{\mathrm{S}_{\tilde{n}}(N)}\right]\\
&\le K_B\mathrm{E}\Bigg\{\Bigg\{\int_0^{T\wedge\theta_N}|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\|\mathbf{v}\||\mathbf{u}_n(s)|^{\frac12}\|\mathbf{u}_n(s)\|^{\frac12}\mathrm{d}s\\
&\quad+\int_0^{T\wedge\theta_N}|\mathbf{u}(s)|^{\frac12}\|\mathbf{u}(s)\|^{\frac12}\|\mathbf{v}\||\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\mathrm{d}s\Bigg\}\chi_{\mathrm{S}_{\tilde{n}}(N)}\Bigg\}\\
&\le K_B\|\mathbf{v}\|\mathrm{E}\Bigg\{\Bigg\{\sup_{t\le T\wedge\theta_N}|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}|\mathbf{u}_n(s)|^{\frac12}\\
&\quad\times\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\|\mathbf{u}_n(s)\|^{\frac12}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{t\le T\wedge\theta_N} |\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}|\mathbf{u}(s)|^{\frac12}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\|\mathbf{u}(s)\|^{\frac12}\mathrm{d}s\Big\}\chi_{S_{\tilde n}(N)}\Big\}\\
&\le K_B\|\mathbf{v}\|\mathrm{E}\Big\{\Big\{\sup_{t\le T\wedge\theta_N}\Big[|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}|\mathbf{u}(s)|^{\frac12}+|\mathbf{u}_n(s)-\mathbf{u}(s)|\Big]\\
&\quad\times\int_0^{T\wedge\theta_N}\Big[\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\|\mathbf{u}_n(s)\|^{\frac12}+\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\Big]\mathrm{d}s\\
&\quad+\sup_{t\le T\wedge\theta_N}|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}|\mathbf{u}(s)|^{\frac12}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\|\mathbf{u}(s)\|^{\frac12}\mathrm{d}s\Big\}\chi_{S_{\tilde n}(N)}\Big\}\\
&\le K_B\|\mathbf{v}\|\mathrm{E}\Big\{\sup_{t\le T\wedge\theta_N}\Big[2|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}|\mathbf{u}(s)|^{\frac12}+|\mathbf{u}_n(s)-\mathbf{u}(s)|\Big]\\
&\quad\times\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\|\mathbf{u}_n(s)\|^{\frac12}\mathrm{d}s\,\chi_{S_{\tilde n}(N)}\Big\}\\
&\quad+K_B\|\mathbf{v}\|\mathrm{E}\Big\{\sup_{t\le T\wedge\theta_N}|\mathbf{u}_n(s)-\mathbf{u}(s)|^{\frac12}|\mathbf{u}_n(s)|^{\frac12}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\,\chi_{S_{\tilde n}(N)}\Big\}\\
&\le 3NK_B\|\mathbf{v}\|\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|^{\frac12}\|\mathbf{u}_n(s)\|^{\frac12}\mathrm{d}s\\
&\quad+2NK_B\|\mathbf{v}\|\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\\
&\le 3NK_B\|\mathbf{v}\|\left(\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\right)^{\frac12}\left(\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)\|\mathrm{d}s\right)^{\frac12}\\
&\quad+2NK_B\|\mathbf{v}\|\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\\
&\le 3NK_B\acute{C}\|\mathbf{v}\|\left(\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\right)^{\frac12}\\
&\quad+2NK_B\|\mathbf{v}\|\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s. \qquad (2.84)
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
&\lim_{n\to\infty}\mathrm{E}\left[\sup_{t\le T}\left|\int_0^{t\wedge\theta_N}(\mathbf{B}(\mathbf{u}_n(s)),\mathbf{v})\mathrm{d}s-\int_0^{t\wedge\theta_N}(\mathbf{B}(\mathbf{u}(s)),\mathbf{v})\mathrm{d}s\right|\chi_{S_{\tilde n}(N)}\right]\\
&\quad\le\lim_{n\to\infty}\Big\{3NK_B\acute{C}\|\mathbf{v}\|\left(\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\right)^{\frac12}\\
&\qquad+2NK_B\|\mathbf{v}\|\mathrm{E}\int_0^{T\wedge\theta_N}\|\mathbf{u}_n(s)-\mathbf{u}(s)\|\mathrm{d}s\Big\}=0. \qquad (2.85)
\end{aligned}
$$

Now consider,

$$
\mathrm{E}\left[\sup_{t\le T}\left|\int_0^{t\wedge\theta_N}(\psi(s,\mathbf{u}_n(s))\mathrm{d}W(s),\mathbf{v})-\int_0^{t\wedge\theta_N}(\psi(s,\mathbf{u}(s))\mathrm{d}W(s),\mathbf{v})\right|\chi_{S_{\tilde n}(N)}\right]
$$

$$
\begin{aligned}
&\leq \sqrt{2}\mathrm{E}\left[\int_0^{T\wedge\theta_N} (\psi(t,\mathbf{u}_n(t)) - \psi(t,\mathbf{u}(t)),\mathbf{v})^2 \mathrm{d}t\chi_{\mathrm{S}_{\tilde{n}}(N)}\right]^{\frac{1}{2}} \\
&\leq \sqrt{2}|\mathbf{v}|\mathrm{E}\left[\int_0^{T\wedge\theta_N} |\psi(t,\mathbf{u}_n(t)) - \psi(t,\mathbf{u}(t))|^2 \mathrm{d}t\chi_{\mathrm{S}_{\tilde{n}}(N)}\right]^{\frac{1}{2}} \\
&\leq \sqrt{2\mathrm{M}_1}|\mathbf{v}|\left[\mathrm{E}\int_0^{T\wedge\theta_N} |\mathbf{u}_n(t) - \mathbf{u}(t)|^2 \mathrm{d}t\chi_{\mathrm{S}_{\tilde{n}}(N)}\right]^{\frac{1}{2}} \\
&\leq \sqrt{2\mathrm{M}_1 N}|\mathbf{v}|\left[\mathrm{E}\int_0^{T\wedge\theta_N} |\mathbf{u}_n(t) - \mathbf{u}(t)| \mathrm{d}t\chi_{\mathrm{S}_{\tilde{n}}(N)}\right]^{\frac{1}{2}} \\
&\leq \sqrt{2\mathrm{M}_1 N}|\mathbf{v}|\left[\mathrm{E}\int_0^{T\wedge\theta_N} \|\mathbf{u}_n(t) - \mathbf{u}(t)\| \mathrm{d}t\right]^{\frac{1}{2}} \to 0, \quad \text{as } n\to\infty. \qquad (2.86)
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&\mathrm{E}\Bigg[\sup_{t\leq T}\Bigg|\int_0^{t\wedge\theta_N}\int_{\mathbb{Z}_n} (\phi(\mathbf{u}_n(s^-),x),\mathbf{v})\widetilde{N}(\mathrm{d}s,\mathrm{d}x) \\
&\quad - \int_0^{t\wedge\theta_N}\int_{\mathbb{H}} (\phi(\mathbf{u}(s^-),x),\mathbf{v})\widetilde{N}(\mathrm{d}s,\mathrm{d}x)\Bigg|\chi_{\mathrm{S}_{\tilde{n}}(N)}\Bigg] \\
&\leq \sqrt{2}|\mathbf{v}|\mathrm{E}\left(\int_0^{T\wedge\theta_N}\int_{\mathbb{Z}_n} |\phi(\mathbf{u}_n(s^-),x) - \phi(\mathbf{u}(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad + \sqrt{2}|\mathbf{v}|\mathrm{E}\left(\int_0^{T\wedge\theta_N}\int_{\mathbb{Z}_n^c} |\phi(\mathbf{u}(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \sqrt{2\mathrm{M}_2}|\mathbf{v}|\left[\mathrm{E}\int_0^{T\wedge\theta_N} |\mathbf{u}_n(t) - \mathbf{u}(t)|^2\mathrm{d}t\right]^{\frac{1}{2}} \\
&\quad + \sqrt{2}|\mathbf{v}|\mathrm{E}\left(\int_0^{T\wedge\theta_N}\int_{\mathbb{Z}_n^c} |\phi(\mathbf{u}(s^-),x)|^2\lambda(\mathrm{d}x)\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \sqrt{2\mathrm{M}_2 N}|\mathbf{v}|\left[\mathrm{E}\int_0^{T} \|\mathbf{u}_n(t) - \mathbf{u}(t)\|\mathrm{d}t\right]^{\frac{1}{2}} \\
&\quad + N\sqrt{T}|\mathbf{v}|\mathrm{E}\left(\sup_{|\mathbf{u}|\leq N}\int_{\mathbb{Z}_n^c} |\phi(\mathbf{u},x)|^2\lambda(\mathrm{d}x)\right)^{\frac{1}{2}} \to 0, \quad \text{as } n\to\infty. \qquad (2.87)
\end{aligned}
$$

From (2.83), (2.85), (2.86), and (2.87) we have,

$$
\begin{aligned}
(\mathbf{u}(t),\mathbf{v}) &= (\mathbf{u}(0),\mathbf{v}) - \nu\int_0^t (\mathbf{A}\mathbf{u}(s),\mathbf{v})\mathrm{d}s - \int_0^t (\mathbf{B}(\mathbf{u}(s)),\mathbf{v})\mathrm{d}s \\
&\quad + \int_0^t (\mathbf{u}(s)\mathrm{dW}(s),\mathbf{v}) + \int_0^t\int_{\mathbb{H}} (\phi(\mathbf{u}(s^-),x),\mathbf{v})\widetilde{N}(\mathrm{d}s,\mathrm{d}x), \qquad (2.88)
\end{aligned}
$$

for $\omega \in \mathrm{S}_{\tilde{n}}(N)$ and $t \in [0,\theta_N)$. Since $\theta_N \to \infty$ as $N \to \infty$ and $\cup_{\tilde{n}=1}^{\infty}\mathrm{S}_{\tilde{n}}(N) = \Omega$, $\mathbf{u}(t)$ is a strong pathwise solution of (2.14).

Let us now prove the uniqueness of the strong pathwise solution of (2.14). Suppose that $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are two strong pathwise solutions of (2.14). Now define $\vartheta_N = \inf\{t > 0 : \int_0^t \|\mathbf{u}(s)\|\mathrm{d}s \vee \int_0^t \|\mathbf{w}(s)\|\mathrm{d}s > N\}$. Following a similar procedure as in the existence part,

$$\begin{aligned}
&\mathrm{E}\left[\sup_{s\le t\wedge\vartheta_N} |\mathbf{u}(s)-\mathbf{w}(s)|^2 + \nu\int_0^{t\wedge\vartheta_N} \|\mathbf{u}(s)-\mathbf{w}(s)\|^2\mathrm{d}s\right] \\
&\le C(N,t)\mathrm{E}\left\{\int_0^t |\psi(s\wedge\vartheta_N, \mathbf{u}(s\wedge\vartheta_N)) - \psi(s\wedge\vartheta_N, \mathbf{w}(s\wedge\vartheta_N))|^2\mathrm{d}s\right. \\
&\quad \left. + \int_0^t\int_{\mathbb{H}} |\phi(\mathbf{u}(s\wedge\vartheta_N -), x) - \phi(\mathbf{w}(s\wedge\vartheta_N -), x)|^2\lambda(\mathrm{d}x)\mathrm{d}s\right\} \\
&\le C(N,t)(\mathrm{M}_1+\mathrm{M}_2)\mathrm{E}\int_0^t \sup_{\acute{s}\le s\wedge\vartheta_N} |\mathbf{u}(\acute{s})-\mathbf{w}(\acute{s})|^2\mathrm{d}s. \qquad (2.89)
\end{aligned}$$

Apply Gronwall's inequality for (2.89):

$$\mathrm{E}\left[\sup_{s\le t\wedge\vartheta_N} |\mathbf{u}(s)-\mathbf{w}(s)|^2 + \nu\int_0^{t\wedge\vartheta_N} \|\mathbf{u}(s)-\mathbf{w}(s)\|^2\mathrm{d}s\right] = 0. \qquad (2.90)$$

From (2.90), we have uniqueness of the strong pathwise solution of (2.14) since $\vartheta_N \to \infty$ as $N \to \infty$. □

3. FKK and Zakai Equations

3.1. *Derivation of FKK and Zakai Equations*

In this section, we estimate the signal process $\{\mathbf{u}(t), 0 \le t \le T\}$ (2.14) using partial (sensor) measurements (see [15, 38, 48] and also [4]). The observation process $\{\mathrm{Z}(t), 0 \le t \le T\}$ is associated with $\mathbf{u}(t)$ as follows:

$$\mathrm{dZ}(t) = h(\mathbf{u}(t))\mathrm{d}t + \mathrm{dB}(t) \qquad (3.1)$$

where,

(C1) $\mathrm{B}(t)$ is a $\mathbb{R}^m$-valued Brownian motion;
(C2) $h : \mathbb{H} \to \mathbb{R}^m$, has atmost quadratic growth rate $|h(\mathbf{u})| \le \mathrm{c}_h(1 + |\mathbf{u}|^2)$ for all $\mathbf{u} \in \mathbb{H}$. For example, $h(\mathbf{u}) = ((\mathbf{u}, e_1), \ldots, (\mathbf{u}, e_m))$, where $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis for $\mathbb{H}_m$(finite-dimensional subspace of $\mathbb{H}$);
(C3) $\mathrm{W}(.)$, $N(.,.)$ and $\mathrm{B}(.)$ are independent.

Note 3.1. In [38], two assumptions are made regarding the observation vector $h(.)$. First assumption is that $h(.)$ is unbounded with quadratic growth. Second assumption is almost similar to C2 except that we have quadratic growth of $h(.)$ involved with $\mathbb{H}$-norm and in [38] the quadratic growth of $h(.)$ is associated with $\mathbb{V}$-norm. In this work, we assume that signal process and observation process are independent. On the other hand, signal and observation processes are correlated in [38].

Let us introduce the notations for a family of σ-fields associated with $\mathbf{u}(t)$, $Z(t)$ and $B(t)$, $0 \le t \le T$:

$$\mathscr{F}_t^Z = \sigma\{Z(s); s \le t\} \quad \text{and} \quad \mathscr{G}_t = \sigma\{\mathbf{u}(s), B(s); s \le t\}.$$

Definition 3.1. Let $\mathbf{u}(t) \in \mathbb{H}$, $t > 0$ be the jump diffusion associated with (2.14) with transition semigroup S_t. Then the infinitesimal generator $\mathscr{A}$ of $f(\mathbf{u}(t))$ with $f : \mathbb{H} \to \mathbb{R}$ is defined as follows:

$$\mathscr{A}f = \lim_{t \downarrow 0^+} \frac{S_t f - f}{t}, \quad \forall f \in D(\mathscr{A}), \tag{3.2}$$

where

$$D(\mathscr{A}) = \left\{ f : \mathbb{H} \to \mathbb{R} \text{ such that } \lim_{t \downarrow 0^+} \frac{S_t f - f}{t} \text{ exists} \right\}. \tag{3.3}$$

Proposition 3.1. *Let $\mathbf{u}(t)$ be a solution of (2.14). Then for $f \in D(\mathscr{A})$, the formal infinitesimal generator $\mathscr{A}f(\mathbf{v})$ is given by*

$$\begin{aligned} \mathscr{A}f(\mathbf{v}) &= -\langle \nu \mathbf{A}\mathbf{v} + \mathbf{B}(\mathbf{v}) - \mathbf{f}, D_{\mathbf{v}} f \rangle + \frac{1}{2} Tr(\psi(\mathbf{v}) Q \psi^*(\mathbf{v}) D_{\mathbf{v}}^2 f) \\ &\quad + \int_{\mathbb{H}} \{ f(\mathbf{v} + \phi(\mathbf{v}, x)) - f(\mathbf{v}) - \langle \phi(\mathbf{v}, x) \chi_{\{|x|<1\}}, D_{\mathbf{v}} f \rangle \} \lambda(\mathrm{d}x), \end{aligned} \tag{3.4}$$

where $\mathbf{v} \in D(\mathbf{A})$.

Proof. Proof of the finite-dimensional counterpart can be found in [35] and Theorem 3.3.3 in [2]. □

Definition 3.2. The class of cylindrical test functions $\mathbb{C}_{\text{cyl}}$ is defined by

$$\begin{aligned} \mathbb{C}_{\text{cyl}} = \{ f : \mathbb{H} \to \mathbb{R};\ & f(\mathbf{u}) = \varphi((\mathbf{u}, e_1), (\mathbf{u}, e_2), \ldots, (\mathbf{u}, e_n)), \\ & e_i \in D(\mathbf{A}), i = 1, \ldots, n; \varphi \in \mathbb{C}_0^\infty(\mathbb{R}^n) \}. \end{aligned} \tag{3.5}$$

Note that $D_u f \in D(\mathbf{A})$ for $f \in \mathbb{C}_{\text{cyl}}$. Rest of the article, we will often work with $f \in \mathbb{C}_{\text{cyl}} \subseteq D(\mathscr{A})$.

The least square best estimate for $f(\mathbf{u}(t))$ given back measurements $Z(s), 0 \le s \le t$, is given by conditional expectation $E[f(\mathbf{u}(t)) \,|\, \mathscr{F}_t^Z]$. This best estimate is the solution of Fujisaki-Kallianpur-Kunita [11] equation (i.e., nonlinear filtering equation). Now we present some results from Fujisaki et al. [11] that are needed for the derivation of FKK and Zakai equations.

Lemma 3.2. *Let $\nu(t) = Z(t) - \int_0^t \hat{h}(\mathbf{u}(s))\mathrm{d}s$, (where $\hat{h}(\mathbf{u}(s)) = E[h(\mathbf{u}(s)) \,|\, \mathscr{F}_s^Z]$) be the innovation process associated with observation process $Z(t)$. Under the assumptions C1–C3 and Theorem 2.3, $(\nu(t), \mathscr{F}_t^Z, P)$ is an $\mathbb{R}^m$-valued standard Wiener process. Furthermore, the sigma algebras $\mathscr{F}_s^Z$ and $\sigma\{\nu(t) - \nu(\tau); s \le \tau < t \le T\}$ are independent.*

Proof. See Lemma 2.2 in [11]. □

Theorem 3.3. *Under assumptions C1–C3 and Theorem 2.3, every separable square integrable martingale* $(\mathrm{Y}(t), \mathscr{F}_t^Z, \mathrm{P})$ *(see [20]) is sample continuous and has the representation*

$$\mathrm{Y}(t) - \mathrm{E}[\mathrm{Y}(0)] = \int_0^t \theta(s) \cdot \mathrm{d}\boldsymbol{v}(s), \tag{3.6}$$

where $\int_0^T \mathrm{E}|\theta(s)|^2 \mathrm{d}s < \infty$ *and* $\theta(s) = (\theta_1(s), \ldots, \theta_m(s))$ *is jointly measurable and adapted to* $\{\mathscr{F}_t^Z\}$.

Proof. See Theorem 3.1 in [11]. □

Lemma 3.4. *Let* $f \in \mathbb{C}_{\underline{\mathrm{cyl}}} \subseteq \mathrm{D}(\mathscr{A})$ *and let* $\overline{\mathrm{M}}_t(f) = \mathrm{E}[f(\mathbf{u}(t)) \mid \mathscr{F}_t^Z] - \mathrm{E}[f(\mathbf{u}(0)) \mid \mathscr{F}_0^Z] - \int_0^t \mathrm{E}[\mathscr{A}f \mid \mathscr{F}_s^Z]\mathrm{d}s$. *Then* $(\overline{\mathrm{M}}_t(f), \mathscr{F}_t^Z, \mathrm{P})$ *is a locally square integrable martingale.*

Proof. See Lemma 4.1 in [11]. □

Theorem 3.5. *Suppose that (2.14), (3.1) and conditions C1–C3 hold. If* $f \in \mathbb{C}_{\mathrm{cyl}} \subseteq \mathrm{D}(\mathscr{A})$ *then* $\mathrm{E}(f(\mathbf{u}(t)) \mid \mathscr{F}_t^Z)$ *satisfy the following stochastic differential equation*

$$\begin{aligned}\mathrm{E}(f \mid \mathscr{F}_t^Z) &= \mathrm{E}(f \mid \mathscr{F}_0^Z) + \int_0^t \mathrm{E}(\mathscr{A}f \mid \mathscr{F}_s^Z)\mathrm{d}s \\ &\quad + \int_0^t \left[\mathrm{E}(fh \mid \mathscr{F}_s^Z) - \mathrm{E}(f \mid \mathscr{F}_s^Z)\mathrm{E}(h \mid \mathscr{F}_s^Z)\right] \cdot \mathrm{d}\boldsymbol{v}(s), \end{aligned} \tag{3.7}$$

where $\boldsymbol{v}(t), 0 \le t \le T$ *is the innovation process.*

Proof. First we show that FKK equation (3.7) is well defined.

By applying Jensen's inequality, Cauchy-Schwartz inequality, growth rate of h, boundedness of f, and Theorem 2.3,

$$\begin{aligned}\mathrm{E}\int_0^t |\mathrm{E}(fh \mid \mathscr{F}_s^Z)|^2 \mathrm{d}s &\le \mathrm{E}\int_0^t \mathrm{E}[|fh|^2|\mathscr{F}_s^Z]\mathrm{d}s \\ &= \int_0^t \mathrm{E}|fh|^2\mathrm{d}s \le \mathrm{c}_h\mathrm{c}_f \int_0^t \mathrm{E}(1 + |\mathbf{u}(s)|^2)^2\mathrm{d}s \\ &\le 2\mathrm{c}_h\mathrm{c}_f\left\{\int_0^t \mathrm{E}|\mathbf{u}(s)|^4\mathrm{d}s + t\right\} < \infty, \quad \text{for } 0 \le t \le T. \end{aligned} \tag{3.8}$$

Using Jensen's inequality, growth rate of h, boundedness of f, we obtain that

$$\mathrm{E}\int_0^t \left|\mathrm{E}(f|\mathscr{F}_s^Z)\mathrm{E}(h|\mathscr{F}_s^Z)\right|^2 \mathrm{d}s \le \mathrm{c}_f \int_0^t \mathrm{E}|\mathrm{E}(h|\mathscr{F}_s^Z)|^2\mathrm{d}s. \tag{3.9}$$

We can show that estimate (3.9) is finite for all $0 \le t \le T$ by a similar procedure as in (3.8) with quadratic growth rate of h. Hence, we have

$$\mathrm{E}\int_0^t \left[\mathrm{E}(fh|\mathscr{F}_s^Z) - \mathrm{E}(f|\mathscr{F}_s^Z)\mathrm{E}(h|\mathscr{F}_s^Z)\right] \cdot \mathrm{d}\boldsymbol{v}(s) = 0, \quad \text{for } 0 \le t \le T. \tag{3.10}$$

Now let us show that

$$\int_0^t \mathrm{E}|\mathrm{E}(\mathscr{A}f|\mathscr{F}_s^Z)|\mathrm{d}s \le \int_0^t \mathrm{E}|\mathscr{A}f|\mathrm{d}s < \infty. \tag{3.11}$$

From Proposition 3.1,

$$\begin{aligned}\int_0^t \mathrm{E}|\mathscr{A}f(\mathbf{u}(s))|\mathrm{d}s \le \int_0^t \mathrm{E}|\left\langle \nu\mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s), D_u f\right\rangle|\mathrm{d}s \\ + \frac{1}{2}\int_0^t \mathrm{E}|\mathrm{Tr}(\psi(\mathbf{u}(s))\mathrm{Q}\psi^*(\mathbf{u}(s))D_u^2 f)|\mathrm{d}s \\ + \int_0^t \mathrm{E}\int_{\mathbb{H}} |f(\mathbf{u}(s) + \phi(\mathbf{u}(s^-), x)) - f(\mathbf{u}(s)) \\ - \left\langle \phi(\mathbf{u}(s^-), x)\chi_{\{|x|<1\}}, D_u f\right\rangle|\lambda(\mathrm{d}x)\mathrm{d}s\end{aligned} \tag{3.12}$$

Using the fact that $\mathbf{A}$ is a self-adjoint operator, $\mathbf{A}D_u f \in \mathbb{H}$ and Theorem 2.3,

$$\begin{aligned}\mathrm{E}\int_0^t |\left\langle \nu\mathbf{A}\mathbf{u}(s), D_u f\right\rangle|\mathrm{d}s &= \nu\mathrm{E}\int_0^t |\left\langle \mathbf{u}(s), \mathbf{A}D_u f\right\rangle|\mathrm{d}s \\ &\le \nu\left(\mathrm{E}|\mathbf{A}D_u f|^2\right)^{\frac{1}{2}}\left(\mathrm{E}\left(\int_0^t |\mathbf{u}(s)|\mathrm{d}s\right)^2\right)^{\frac{1}{2}} \\ &\le \nu\left(t\mathrm{E}|\mathbf{A}D_u f|^2\right)^{\frac{1}{2}}\mathrm{E}\int_0^t |\mathbf{u}(s)|^2\mathrm{d}s < \infty.\end{aligned} \tag{3.13}$$

By using the estimation for nonlinear operator in Section 2.3, [46] and Theorem 2.3,

$$\begin{aligned}&\mathrm{E}\int_0^t |\left\langle \mathbf{B}(\mathbf{u}(s)), D_u f\right\rangle|\mathrm{d}s \\ &\le \mathrm{E}\int_0^t |\mathbf{u}(s)|\|\mathbf{u}(s)\||D_u f|^{\frac{1}{2}}|\mathbf{A}D_u f|^{\frac{1}{2}}\mathrm{d}s \le \left(\mathrm{E}|\mathbf{A}D_u f|^2\right)^{\frac{1}{2}}\left(\mathrm{E}\left(\int_0^t |\mathbf{u}(s)|\|\mathbf{u}(s)\|\mathrm{d}s\right)^2\right)^{\frac{1}{2}} \\ &\le \left(t\mathrm{E}|\mathbf{A}D_u f|^2\right)^{\frac{1}{2}}\mathrm{E}\int_0^t |\mathbf{u}(s)|^2\|\mathbf{u}(s)\|^2\mathrm{d}s < \infty.\end{aligned} \tag{3.14}$$

Since $\mathbf{f} \in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{V}'))$ and $D_u f \in \mathrm{D}(\mathbf{A})$,

$$\begin{aligned}\mathrm{E}\int_0^t |\left\langle \mathbf{f}(s), D_u f\right\rangle|\mathrm{d}s &\le \mathrm{E}\left(\|D_u f\|\int_0^t \|\mathbf{f}(s)\|_{\mathbb{V}'}\mathrm{d}s\right) \\ &\le \left(t\mathrm{E}\|D_u f\|^2\right)^{\frac{1}{2}}\mathrm{E}\int_0^t \|\mathbf{f}(s)\|_{\mathbb{V}'}^2\mathrm{d}s < \infty.\end{aligned} \tag{3.15}$$

Consider the integrand of the second term that appears in right-hand side of (3.12).

$$\begin{aligned}&\mathrm{E}\int_0^t |\mathrm{Tr}(\psi(\mathbf{u}(s))\mathrm{Q}\psi^*(\mathbf{u}(s))D_u^2 f)|\mathrm{d}s \\ &= \mathrm{E}\int_0^t \left|\sum_{i=1}^\infty \left\langle \psi(\mathbf{u}(s))\mathrm{Q}\psi^*(\mathbf{u}(s))D_u^2 f e_i, e_i\right\rangle\right|\mathrm{d}s \\ &= \mathrm{E}\int_0^t \left|\sum_{i=1}^\infty \left\langle \psi(\mathbf{u}(s))\mathrm{Q}\psi^*(\mathbf{u}(s))\left(\sum_{k=1}^n\sum_{l=1}^n \partial_{kl}\varphi((\mathbf{u}, e_1), \dots, (\mathbf{u}, e_n))e_k \otimes e_l\right)e_i, e_i\right\rangle\right|\mathrm{d}s\end{aligned}$$

$$= \mathrm{E}\int_0^t \left| \sum_{k,l=1}^{n} \partial_{kl}\varphi((\mathbf{u}, e_1), \ldots, (\mathbf{u}, e_n)) \left\langle \psi(\mathbf{u}(s))\mathrm{Q}\psi^*(\mathbf{u}(s))e_k, e_l \right\rangle \right| \mathrm{d}s$$

$$\leq \mathrm{E}\left(\|D_u^2 f\| \int_0^t \|\psi(\mathbf{u}(s))\|_{\mathrm{L_Q}} \mathrm{d}s \right) \leq \left(\mathrm{E}\|D_u^2 f\|^2\right)^{\frac{1}{2}} \left(\mathrm{E}\left(\int_0^t \|\psi(\mathbf{u}(s))\|_{\mathrm{L_Q}} \mathrm{d}s \right)^2 \right)^{\frac{1}{2}}$$

$$\leq \left(t\mathrm{E}\|D_u^2 f\|^2\right)^{\frac{1}{2}} \mathrm{E}\int_0^t \|\psi(\mathbf{u}(s))\|_{\mathrm{L_Q}}^2 \mathrm{d}s$$

$$\leq \mathrm{K}_1 \left(t\mathrm{E}\|D_u^2 f\|^2\right)^{\frac{1}{2}} \mathrm{E}\int_0^t (1 + |\mathbf{u}(s)|^2 \mathrm{d}s) < \infty, \tag{3.16}$$

where the complete orthonormal system $\{e_1, e_2, \ldots, e_n, \ldots\} \subseteq \mathrm{D}(\mathbf{A})$; condition A1, Theorem 2.3 and $D_u^2 f \in \mathrm{D}(\mathbf{A})$ are used in deriving (3.16).

Applying Taylor's theorem for f with integral remainder term (see Theorem 4.4.7 in [2]), we find that

$$\begin{aligned} &\int_0^t \mathrm{E}\int_{|x|<1} \left| f(\mathbf{u}(s) + \phi(\mathbf{u}(s^-), x)) - f(\mathbf{u}(s)) - \langle \phi(\mathbf{u}(s^-), x), D_u f \rangle \right| \lambda(\mathrm{d}x)\mathrm{d}s \\ &\quad = \int_0^t \mathrm{E}\int_{|x|<1} \int_0^1 \sum_{k,l=1}^{n} \partial_{kl}\varphi((\mathbf{u} + \theta\phi, e_1), \ldots, (\mathbf{u} + \theta\phi, e_n))(1-\theta)\mathrm{d}\theta \\ &\quad\quad \times \langle \phi(\mathbf{u}(s^-), x), e_k \rangle \langle \phi(\mathbf{u}(s^-), x), e_l \rangle \lambda(\mathrm{d}x)\mathrm{d}s. \end{aligned} \tag{3.17}$$

For each $t \geq 0$, $0 < |x| < 1$ and $1 \leq k, l \leq n$, define

$$g_{k,l}^{\varphi}(t, x) = \sup_{0\leq\theta\leq1} \partial_{kl}\varphi((\mathbf{u} + \theta\phi, e_1), \ldots, (\mathbf{u} + \theta\phi, e_n)).$$

Since f is a cylindrical test function, we get,

$$\sup_{0\leq s\leq t}\ \sup_{0<|x|<1} |g_{k,l}^{\varphi}(t, x)| < \infty \text{ a.s.} \tag{3.18}$$

Applying the Cauchy-Schwartz inequality for (3.17), we obtain

$$\begin{aligned} &\int_0^t \mathrm{E}\int_{|x|<1} \left| f(\mathbf{u}(s) + \phi(\mathbf{u}(s^-), x)) - f(\mathbf{u}(s)) - \langle \phi(\mathbf{u}(s^-), x), D_u f \rangle \right| \lambda(\mathrm{d}x)\mathrm{d}s \\ &\quad \leq \mathrm{E}\Bigg(\frac{1}{2} \sum_{k,l=1}^{n} \sup_{0\leq s\leq t}\ \sup_{0<|x|<1} |g_{k,l}^{\varphi}(t, x)| \\ &\quad\quad \times \int_0^t \int_{|x|<1} \left| \langle \phi(\mathbf{u}(s^-), x), e_k \rangle \langle \phi(\mathbf{u}(s^-), x), e_l \rangle \right| \lambda(\mathrm{d}x)\mathrm{d}s \Bigg) \\ &\quad \leq \frac{1}{2}\mathrm{E}\left(\sum_{k,l=1}^{n} \sup_{0\leq s\leq t}\ \sup_{0<|x|<1} |g_{k,l}^{\varphi}(t, x)|^2 \right)^{\frac{1}{2}} \left(\mathrm{E}\int_0^t \int_{|x|<1} |\phi(\mathbf{u}(s^-), x)|^2 \lambda(\mathrm{d}x)\mathrm{d}s \right)^{\frac{1}{2}} \\ &\quad \leq \frac{\mathrm{K}_2}{2}\mathrm{E}\left(t\sum_{k,l=1}^{n} \sup_{0\leq s\leq t}\ \sup_{0<|x|<1} |g_{k,l}^{\varphi}(t, x)|^2 \right)^{\frac{1}{2}} \left(\mathrm{E}\int_0^t \left(1 + |\mathbf{u}(s)|^2\right)^2 \mathrm{d}s \right)^{\frac{1}{2}} < \infty. \end{aligned} \tag{3.19}$$

The last inequality holds due to (3.18) , condition B1, and Theorem 2.3.

By combining (3.13), (3.14), (3.16), (3.19) with the inequality $\int_0^t \mathrm{E}|\mathrm{E}(\mathscr{A}f|\mathscr{F}_s^Z)|\mathrm{d}s \leq \int_0^t \mathrm{E}|\mathscr{A}f|\mathrm{d}s$, we obtain (3.11).

Then (3.10) and (3.11) immediately give us $\mathrm{E}[\mathrm{E}(f|\mathscr{F}_t^Z)] < \infty$.

Let us now derive the FKK equation (3.7). Let

$$\mathrm{M}_t^*(f) = \int_0^t \left[\mathrm{E}(fh|\mathscr{F}_s^Z) - \mathrm{E}(f|\mathscr{F}_s^Z)\mathrm{E}(h|\mathscr{F}_s^Z)\right] \cdot \mathrm{d}\boldsymbol{v}(s). \tag{3.20}$$

Then (3.7) reduce to

$$\mathrm{M}_t^*(f) = \overline{\mathrm{M}}_t(f), \tag{3.21}$$

where $\overline{\mathrm{M}}_t(f) = \mathrm{E}(f|\mathscr{F}_t^Z) - \mathrm{E}(f|\mathscr{F}_0^Z) - \int_0^t \mathrm{E}(\mathscr{A}f|\mathscr{F}_s^Z)\mathrm{d}s$. Then (3.7) is proven if

$$\mathrm{E}\left[(\mathrm{M}_t^*(f) - \overline{\mathrm{M}}_t(f))\mathrm{Y}(t)\right] = 0, \tag{3.22}$$

for all Y_t such that $\mathrm{Y}(t) = \int_0^t \theta(s) \cdot \mathrm{d}\boldsymbol{v}(s)$ and dense in $\mathbb{L}^2(\mathscr{F}_t^Z, \mathrm{P})$.

To prove (3.22), we determine $\mathrm{E}[(\mathrm{M}_t(f) - \overline{\mathrm{M}}_t(f))\mathrm{Y}(t)]$ and $\mathrm{E}[\mathrm{M}_t(f)\mathrm{Y}(t)]$ separately, where $\mathrm{M}_t(f) = f(\mathbf{u}(t)) - f(\mathbf{u}(0)) - \int_0^t \mathscr{A}f\mathrm{d}s$ is a $(\mathscr{G}_t, \mathrm{P})$ martingale.

Using the fact that $\mathrm{Y}(t)$ is a square integrable martingale adapted to $\mathscr{F}_t^Z$,

$$\begin{aligned}
&\mathrm{E}\left[(\mathrm{M}_t(f) - \overline{\mathrm{M}}_t(f))\mathrm{Y}(t)\right] \\
&= \mathrm{E}\left\{\mathrm{Y}(t)\int_0^t \left[\mathscr{A}f - \mathrm{E}(\mathscr{A}f|\mathscr{F}_s^Z)\right]\mathrm{d}s\right\} \\
&= \mathrm{E}\int_0^t \left[\mathrm{Y}(t)\mathscr{A}f - (\mathrm{Y}(t) - \mathrm{Y}(s))\mathrm{E}(\mathscr{A}f|\mathscr{F}_s^Z) - \mathrm{Y}(s)\mathrm{E}(\mathscr{A}f|\mathscr{F}_s^Z)\right]\mathrm{d}s \\
&= \int_0^t \{\mathrm{E}\left[(\mathrm{Y}(t) - \mathrm{Y}(s))\mathscr{A}f\right] - \mathrm{E}(\mathrm{Y}(t) - \mathrm{Y}(s))\mathrm{E}(\mathscr{A}f)\}\,\mathrm{d}s \\
&= \mathrm{E}\int_0^t (\mathrm{Y}(t) - \mathrm{Y}(s))(\mathscr{A}f)\mathrm{d}s
\end{aligned} \tag{3.23}$$

Last, equality holds since $\mathrm{Y}(t) - \mathrm{Y}(s)$ and $\mathrm{E}(\mathscr{A}f|\mathscr{F}_s^Z)$ are independent and $\mathrm{E}(\mathrm{Y}(t) - \mathrm{Y}(s)) = 0$.

Substituting $\mathrm{Y}(t) = \int_0^t \theta(s) \cdot \mathrm{dB}(s) + \int_0^t \theta(s) \cdot (h_s - \hat{h}_s)\mathrm{d}s$ into right hand side of (3.23),

$$\begin{aligned}
\mathrm{E}\left[(\mathrm{M}_t(f) - \overline{\mathrm{M}}_t(f))\mathrm{Y}(t)\right] &= \mathrm{E}\int_0^t \left[(\mathscr{A}f)\int_s^t \theta(\tau) \cdot \mathrm{dB}(\tau)\right]\mathrm{d}s \\
&\quad + \mathrm{E}\int_0^t \left[(\mathscr{A}f)\int_s^t \theta(\tau) \cdot (h_\tau - \hat{h}_\tau)\mathrm{d}\tau\right]\mathrm{d}s.
\end{aligned} \tag{3.24}$$

First term of right-hand side of (3.24) coincide with

$$\mathrm{E}\int_0^t \left((\mathscr{A}f)\mathrm{E}\left[\int_s^t \theta(\tau) \cdot \mathrm{dB}(\tau)\middle|\mathscr{G}_s\right]\right)\mathrm{d}s = 0, \tag{3.25}$$

since $\mathscr{A}f$ is $\mathscr{G}_s$-measurable and $\mathrm{E}[\int_s^t \theta(\tau) \cdot \mathrm{dB}(\tau)\big|\mathscr{G}_s] = 0$.

Integration by parts applied to the second term of the right-hand side of (3.24) gives

$$\mathrm{E}\left[(\mathrm{M}_t(f) - \overline{\mathrm{M}}_t(f))\mathrm{Y}(t)\right] - \mathrm{E}\int_0^t \left[\int_0^s (\mathscr{A}f)\mathrm{d}\tau\right] \theta(s) \cdot (h_s - \hat{h}_s)\mathrm{d}s. \tag{3.26}$$

By contrast, one can easily see that

$$\begin{aligned}\mathrm{E}\,[\mathrm{M}_t(f)\mathrm{Y}(t)] = \mathrm{E}\left[\mathrm{M}_t(f)\int_0^t \theta(s)\cdot \mathrm{dB}(s)\right] + \mathrm{E}\left[f(\mathbf{u}(t))\int_0^t \theta(s)\cdot(h_s - \hat{h}_s)\mathrm{d}s\right]\\ - \mathrm{E}\left[\mathrm{E}(f|\mathscr{F}_0^Z)\int_0^t \theta(s)\cdot(h_s-\hat{h}_s)\mathrm{d}s\right]\\ - \mathrm{E}\left[\int_0^t (\mathscr{A}f)\mathrm{d}s\int_0^t \theta(s)\cdot(h_s-\hat{h}_s)\mathrm{d}s\right].\end{aligned} \tag{3.27}$$

The first term that appears in the right-hand side of (3.27) is zero due to condition C3 and the fact that $\int_0^t \theta(s)\cdot \mathrm{dB}(s)$ is a martingale. We can easily show that the third term of the right hand side of (3.27) is zero by taking to account that $\mathbb{E}(f\,|\,\mathscr{F}_0^2)$ and $\theta(s)$ are $\mathscr{F}_s^Z$-measurable and $\theta(s)\cdot \mathrm{E}[h_s - \hat{h}_s\,|\,\mathscr{F}_s^Z] = 0$. Then (3.27) reduces to

$$\begin{aligned}\mathrm{E}\,[\mathrm{M}_t(f)\mathrm{Y}(t)] &= \mathrm{E}\left[f(\mathbf{u}(t))\int_0^t \theta(s)\cdot(h_s-\hat{h}_s)\mathrm{d}s\right]\\ &\quad - \mathrm{E}\left[\int_0^t(\mathscr{A}f)\mathrm{d}s\int_0^t\theta(s)\cdot(h_s-\hat{h}_s)\mathrm{d}s\right]\\ &= \mathrm{E}\left[\int_0^t f(\mathbf{u}(s))\theta(s)\cdot(h_s-\hat{h}_s)\mathrm{d}s\right]\\ &\quad - \mathrm{E}\left[\int_0^t\left(\int_0^s(\mathscr{A}f)\mathrm{d}\tau\right)\theta(s)\cdot(h_s-\hat{h}_s)\mathrm{d}s\right].\end{aligned} \tag{3.28}$$

Since $(f(\mathbf{u}(t)) - f(\mathbf{u}(s)))$ and $\int_0^s(\mathscr{A}f)\mathrm{d}\tau$ are independent of $\mathscr{F}_s^Z$, $\theta(s)$ is $\mathscr{F}_s^Z$-measurable and $\theta(s)\cdot \mathrm{E}[h_s - \hat{h}_s\,|\,\mathscr{F}_s^Z] = 0$.

Applying properties of stochastic integrals with Equations (3.26)–(3.28) gives

$$\begin{aligned}\mathrm{E}\,[\mathrm{M}_t(f)\mathrm{Y}(t)] &= \mathrm{E}\left[\int_0^t \theta(s)\cdot \mathrm{E}(f(\mathbf{u}(s))(h_s-\hat{h}_s)\,|\,\mathscr{F}_s^Z)\mathrm{d}s\right]\\ &\quad + \mathrm{E}\left[(\mathrm{M}_t(f)-\overline{\mathrm{M}}_t(f))\mathrm{Y}(t)\right]\\ &= \mathrm{E}\left[\mathrm{Y}(t)\int_0^t \mathrm{E}(f(\mathbf{u}(s))(h_s-\hat{h}_s)\,|\,\mathscr{F}_s^Z)\cdot \mathrm{d}\boldsymbol{\nu}(s)\right]\\ &\quad + \mathrm{E}\left[(\mathrm{M}_t(f)-\overline{\mathrm{M}}_t(f))\mathrm{Y}(t)\right].\end{aligned} \tag{3.29}$$

Then (3.20) and (3.29) immediately give us (3.21). □

Now we are interested in deriving Zakai equation by using Girsanov transformation and Kallianpur-Striebel's formula.

Lemma 3.6 (Girsanov transformation). *Assume that the conditions C2, C3, and Theorem 2.3 hold. Then,*

1. *The stochastic process*

$$\alpha_t = \exp\left[-\int_0^t h^i(\mathbf{u}(s))\mathrm{dB}^i(s) - \frac{1}{2}\int_0^t |h(\mathbf{u}(s))|^2\mathrm{d}s\right] \tag{3.30}$$

is a $\mathcal{G}_t$-martingale and $\mathrm{E}[\alpha_t] = 1$ *for all* $0 < t \leq T$.

2. *The measure* $\widetilde{\mathrm{P}}$ *defined by* $\mathrm{d}\widetilde{\mathrm{P}} = \alpha_T \mathrm{dP}$ *is a probability measure and the process*

$$\mathrm{Z}(t) = \mathrm{Z}(0) + \mathrm{B}(t) + \int_0^t h(\mathbf{u}(s))\mathrm{d}s \tag{3.31}$$

is a Weiner process with respect to probability space $(\Omega, \mathcal{G}_t, \widetilde{\mathrm{P}})$.

Proof. All the details of this Lemma follows from Lemma 3.1 in [11] except the result $\mathrm{E}[\alpha_t] = 1$, since h is unbounded function.

In order to obtain $\mathrm{E}[\alpha_t] = 1$ for all $0 < t \leq T$, we use truncation function approach by Ferrario [10].

We define truncation function χ^N for each $N = 1, 2, 3, \ldots$ as follows.

$$\chi_t^N(\mathbf{v}) = \begin{cases} 1 & \text{if } \int_0^t |h(\mathbf{v}(s))|^2\mathrm{d}s \leq N \\ 0 & \text{otherwise} \end{cases} \tag{3.32}$$

From C2, Theorem 2.3 and (3.32), we have Novikov condition

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^t \left|\chi_t^N(\mathbf{u}(s))h(\mathbf{u}(s))\right|^2 \mathrm{d}s\right)\right] < \infty, \tag{3.33}$$

for all $N = 1, 2, 3, \ldots$.
This implies $\mathrm{E}[\alpha_t^N] = 1$ for all $N = 1, 2, 3, \ldots$ with

$$\alpha_t^N = \exp\left[-\int_0^t \chi_t^N(\mathbf{u}(s))h^i(\mathbf{u}(s))\mathrm{dB}^i(s) - \frac{1}{2}\int_0^t \chi_t^N(\mathbf{u}(s))|h(\mathbf{u}(s))|^2\mathrm{d}s\right]. \tag{3.34}$$

Now we prove $\mathrm{E}[\alpha_t] = 1$. Consider

$$\begin{aligned} 1 = \mathrm{E}\left[\alpha_t^N\right] &= \mathrm{E}\left[\chi_t^N(\mathbf{u}(t))\alpha_t^N\right] + \mathrm{E}\left[\left(1 - \chi_t^N(\mathbf{u}(t))\right)\alpha_t^N\right] \\ &= \mathrm{E}\left[\chi_t^N(\mathbf{u}(t))\alpha_t\right] + \mathrm{P}\left\{\chi_t^N(\mathbf{u}(t)) = 0\right\} \end{aligned} \tag{3.35}$$

By applying Monotone convergence theorem, we have

$$\lim_{N\to\infty} \mathrm{E}\left[\chi_t^N(\mathbf{u}(t))\alpha_t\right] = \mathrm{E}\left[\alpha_t\right]. \tag{3.36}$$

On the other hand,

$$\lim_{N\to\infty} \mathrm{P}\left\{\chi_t^N(\mathbf{u}(t)) = 0\right\} = \lim_{N\to\infty} \mathrm{P}\left\{\int_0^t |h(\mathbf{u}(s))|^2\mathrm{d}s > N\right\} = 0. \tag{3.37}$$

The last equality of (3.37) holds by Chebychev's inequality, Theorem 2.3, and condition C2.

Hence by letting $N \to \infty$ in (3.35), we have the result $\mathrm{E}[\alpha_t] = 1$. □

One can prove that under the probability measure $\widetilde{\mathrm{P}}$, processes $\mathrm{Z}(t)$ and $\mathbf{u}(t)$ are independent since condition C3 holds.

Now define

$$\tilde{\alpha}_t = \frac{1}{\alpha_t} = \exp\left[\int_0^t h^i(\mathbf{u}(s))\mathrm{d}Z^i(s) - \frac{1}{2}\int_0^t |h(\mathbf{u}(s))|^2 \mathrm{d}s\right] \tag{3.38}$$

Theorem 3.7. *Suppose that* (2.14), (3.1), *and conditions* C1–C3 *hold. If* f *is a cylindrical test function and* $f \in \mathrm{D}(\mathscr{A})$, *then unnormalized conditional density* $\widetilde{\mathrm{E}}[f(\mathbf{u}(t))\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z]$ *satisfy the following stochastic differential equation*

$$\widetilde{\mathrm{E}}[f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z] = \widetilde{\mathrm{E}}_0(f\tilde{\alpha}_0 \,|\, \mathscr{F}_0^Z) + \int_0^t \tilde{\mathrm{E}}(\mathscr{A}f\tilde{\alpha}_s \,|\, \mathscr{F}_s^Z)\mathrm{d}s + \int_0^t \tilde{\mathrm{E}}(fh\tilde{\alpha}_s \,|\, \mathscr{F}_s^Z)\cdot \mathrm{d}Z(s), \tag{3.39}$$

where $\mathrm{Z}(t)$, $(0 \le t \le T)$ *is the observation process.*

Proof. Applying Itô formula to (3.38), we notice that

$$\tilde{\alpha}_t = 1 + \int_0^t \tilde{\alpha}_s h(\mathbf{u}(s)).\mathrm{d}Z(s). \tag{3.40}$$

Now we take conditional expectation under $\tilde{\mathrm{P}}$ on both sides of (3.40) with respect to $\mathscr{F}_t^Z$, using the fact that $\mathrm{Z}(t)$ is measurable on $\mathscr{F}_t^Z$ and by the Lemma 1.2 on page 102 [5]:

$$\tilde{\mathrm{E}}\left[\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z\right] = 1 + \tilde{\mathrm{E}}\left[\int_0^t \tilde{\alpha}_s h(\mathbf{u}(s)) \,|\, \mathscr{F}_s^Z\right].\mathrm{d}Z(s) \tag{3.41}$$

Then we have,

$$\mathrm{d}\tilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z) = \tilde{\mathrm{E}}(h\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{d}Z(t). \tag{3.42}$$

From Kallianpur-Striebel's formula (see Theorem 3, [18]) ($\mathrm{E}(f \,|\, \mathscr{F}_t^Z) = \tilde{\mathrm{E}}(f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)/\tilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)$) we get

$$\mathrm{d}\tilde{\mathrm{E}}(f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z) = \tilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{d}\mathrm{E}(f \,|\, \mathscr{F}_t^Z) + \mathrm{E}(f \,|\, \mathscr{F}_t^Z)\mathrm{d}\tilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z) + \mathrm{d}\mathrm{E}(f \,|\, \mathscr{F}_t^Z)\cdot \mathrm{d}\tilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z). \tag{3.43}$$

From (3.1), (3.7), (3.42), (3.43), and Kallianpur-Striebel's formula, we obtain

$$\begin{aligned}
&\mathrm{d}\widetilde{\mathrm{E}}(f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z) \\
&\quad = \widetilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\big(\mathrm{E}(\mathscr{A}f \,|\, \mathscr{F}_t^Z)\mathrm{d}t + \big(\mathrm{E}(fh \,|\, \mathscr{F}_t^Z) - \mathrm{E}(f \,|\, \mathscr{F}_t^Z)\mathrm{E}(h \,|\, \mathscr{F}_t^Z)\big)\cdot(\mathrm{d}Z(t) - \hat{h}_t\mathrm{d}t)\big) \\
&\qquad + \mathrm{E}(f \,|\, \mathscr{F}_t^Z)\tilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{E}(h \,|\, \mathscr{F}_t^Z)\cdot \mathrm{d}Z(t)
\end{aligned}$$

$$
\begin{aligned}
&+ (\mathrm{E}(\mathscr{A}f \,|\, \mathscr{F}_t^Z)\mathrm{d}t + (\mathrm{E}(fh \,|\, \mathscr{F}_t^Z) - \mathrm{E}(f \,|\, \mathscr{F}_t^Z)\mathrm{E}(h \,|\, \mathscr{F}_t^Z)) \\
&\cdot (\mathrm{dZ}(t) - \hat{h}_t \mathrm{d}t))\tilde{\mathrm{E}}(\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{E}(h \,|\, \mathscr{F}_t^Z) \cdot \mathrm{dZ}(t) \\
&= \tilde{\mathrm{E}}(\mathscr{A}f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{d}t + (\tilde{\mathrm{E}}(fh\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z) - \tilde{\mathrm{E}}(f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{E}(h \,|\, \mathscr{F}_t^Z)) \cdot (\mathrm{dZ}(t) - \hat{h}_t \mathrm{d}t) \\
&+ \tilde{\mathrm{E}}(h\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{E}(f \,|\, \mathscr{F}_t^Z) \cdot \mathrm{dZ}(t) \\
&+ (\tilde{\mathrm{E}}(fh\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z) - \tilde{\mathrm{E}}(f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{E}(h \,|\, \mathscr{F}_t^Z)) \cdot \hat{h}_t \mathrm{d}t \\
&= \tilde{\mathrm{E}}(\mathscr{A}f\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z)\mathrm{d}t + \tilde{\mathrm{E}}(fh\tilde{\alpha}_t \,|\, \mathscr{F}_t^Z) \cdot \mathrm{dZ}(t). \qquad (3.44)
\end{aligned}
$$

We have, thus, derived the Zakai equation (3.39). □

3.2. *Existence and Uniqueness of Nonlinear Filtering Equations*

Set of all σ-additive finite measures over the Borel sets of $\mathbb{H}$, endowed by weak topology is denoted by $\mathscr{M}(\mathbb{H})$. $\mathscr{M}_+(\mathbb{H})$ and $\mathscr{P}(\mathbb{H})$ are subspaces of $\mathscr{M}(\mathbb{H})$ consisting of all positive measures and probability measures over the Borel sets of $\mathbb{H}$, respectively. If $\mu \in \mathscr{M}(\mathbb{H})$, we represent $\mu(f) := \int_{\mathbb{H}} f(\mathbf{u})\mu(\mathrm{d}\mathbf{u})$ for $f \in \mathbb{C}_{\mathrm{cyl}}$. Now we will define measure-valued solutions for the FKK and Zakai equations.

Definition 3.3. A $\mathscr{P}(\mathbb{H})$-valued process Π_t is called a measure-valued solution of the FKK equation on [0,T] if the following conditions satisfy.

1. Π_t is $\mathscr{F}_t^Z$-measurable for all $0 \le t \le T$.
2.
$$\sup_{0\le t\le T} \mathrm{E}\int_{\mathbb{H}} |\mathbf{v}|^4 \Pi_t(\mathrm{d}\mathbf{v}) + \mathrm{E}\int_0^T \int_{\mathbb{V}} |\mathbf{v}|^2 \|\mathbf{v}\|^2 \Pi_t(\mathrm{d}\mathbf{v})\mathrm{d}t < \infty.$$
3. For all $f \in \mathbb{C}_{\mathrm{cyl}}$ and $0 \le t \le T$ the weak FKK equation holds:

$$\Pi_t(f) = \Pi_0(f) + \int_0^t \Pi_s(\mathscr{A}f)\,\mathrm{d}s + \int_0^t [\Pi_s(hf) - \Pi_s(h)\,\Pi_s(f)] \cdot \mathrm{d}\boldsymbol{\nu}(s), \qquad (3.45)$$

Definition 3.4. A $\mathscr{M}_+(\mathbb{H})$-valued process π_t is called a measure-valued solution of the Zakai equation on [0,T] if the following conditions hold:

1. π_t is $\mathscr{F}_t^Z$-measurable for all $0 \le t \le T$.
2.
$$\sup_{0\le t\le T} \mathrm{E}\int_{\mathbb{H}} |\mathbf{v}|^4 \pi_t(\mathrm{d}\mathbf{v}) + \mathrm{E}\int_0^T \int_{\mathbb{V}} |\mathbf{v}|^2 \|\mathbf{v}\|^2 \pi_t(\mathrm{d}\mathbf{v})\mathrm{d}t < \infty.$$
3. $\mathrm{E}|\pi_t(1)|^2 < \infty$ for all $0 \le t \le T$.
4. $\mathrm{E}\int_0^T |\pi_t(1)|^2 \mathrm{d}t < \infty$
5. For all $f \in \mathbb{C}_{\mathrm{cyl}}$ and $0 \le t \le T$ the weak Zakai equation holds

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(\mathscr{A}f)\mathrm{d}s + \int_0^t \pi_s(hf) \cdot \mathrm{dZ}(s), \qquad (3.46)$$

Note 3.2. By defining,

$$\pi_t(f) = \exp\left\{\int_0^t \Pi_s(h)\cdot \mathrm{d}Z(s) - \frac{1}{2}\int_0^t |\Pi_s(h)|^2 \mathrm{d}s\right\}\Pi_t(f), \tag{3.47}$$

and applying Itô formula to (3.47), one can obtain the measure-valued version of the Zakai equation (3.46), (see Section 3.1 in [38]).

Let us now discuss existence and uniqueness of measure-valued solutions for the FKK and Zakai equations (see [3, 30, 32]).

3.2.1. Existence of Measure-Valued Solutions.

Theorem 3.8. *Assume that the conditions* C1–C3 *and Theorem* 2.3 *hold. Then there exists a measure-valued solution* Π_t *of the FKK equation* (3.45) *on* $[0, T]$ *and there exists a measure-valued solution* π_t *of the Zakai equation* (3.46) *on* $[0, T]$.

Proof. Since Π_t and π_t are related by (3.47), it is enough to prove the existence of one of the two measures. It will be convenient to prove existence for Π_t. We use the lemma 3.9 (R. K. Getoor [13]) to show the existence of Π_t. □

Lemma 3.9 ([13, Proposition 4.1])**.** *Let* $\mathbb{Y}$ *be a Lusin space. We denote by* $B_b(\mathbb{Y})$ *and* $B_b(\Omega)$ *the set of all bounded Borel functions on* $\mathbb{Y}$ *and* Ω *respectively. Now suppose that* $\Psi : B_b(\mathbb{Y}) \to B_b(\Omega)$ *is linear a.e., positive a.e., and satisfies* $0 \le g_n \uparrow g$ *implies that* $\Psi g_n \uparrow \Psi g$ *for any sequence of functions* $\{g_n\}$ *and* $g \in B_b(\mathbb{Y})$*. Then there exist a bounded kernel* $\gamma(.,.)$ *from* $(\Omega, \mathscr{B})$ *to* $(\mathbb{Y}, B(\mathbb{Y}))$ *such that* $\Psi g(\omega) = \int_{\mathbb{Y}} g(\mathbf{v})\gamma(\omega, \mathrm{d}\mathbf{v})$ *for all* $g \in B_b(\mathbb{Y})$ *and* $\omega \in \Omega$.

By using Lemma 3.9, we can find a kernel which is a candidate for the measure that we wish to obtain. Hilbert space $\mathbb{H}$ is a Lusin space since every complete, separable, metric space (Polish space) is a Lusin space. Now define operator: Ψ: $B_b(\mathbb{H}) \to B_b(\Omega)$ by

$$\Psi_t[f](\omega) = \mathrm{E}[f(\mathbf{u}(t)) \mid \mathscr{F}_t^Z], \quad \text{for all } f \in B_b(\mathbb{H}). \tag{3.48}$$

We can easily show that Ψ is a linear, positive, continuous (see Lemma 3.9) operator, a.s.. Then, there exists a $\mathscr{F}_t^Z$-measurable random measure $\Pi(.,.)$ such that

$$\Psi_t[f](\omega) = \int_{\mathbb{H}} f(\mathbf{v})\Pi_t(\omega, \mathrm{d}\mathbf{v}), \quad \text{for all } f \in B_b(\mathbb{H}). \tag{3.49}$$

We then write:

$$\Pi_t(f) = \mathrm{E}[f(\mathbf{u}(t)) \mid \mathscr{F}_t^Z] = \int_{\mathbb{H}} f(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v}) \tag{3.50}$$

for all bounded Borel functions f.

Now we have to check that Π_t is a measure-valued solution of the FKK equation. Definition of the kernel in the lemma implies that Π_t is $\mathscr{F}_t^Z$ measurable. Then we have Condition (1) of Definition 3.2.

Now we want to show that measure-valued solution Π_t satisfies the Condition (2) of the Definition 3.2. Let $g_n(\mathbf{v}) = |\mathbf{v}|^4 \wedge n$ for $n \in \mathbb{N}$. Consider

$$\mathrm{E}\int_{\mathbb{H}} g_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v}) = \mathrm{E}[g_n(\mathbf{u}(t))]. \tag{3.51}$$

The above result holds since $g_n(\mathbf{v})$ is a nonnegative, increasing, bounded Borel function and (3.50). Now take liminf on both sides and apply the monotone convergence theorem for the right-hand side of (3.51) to get,

$$\liminf_{n\to\infty} \mathrm{E}\int_{\mathbb{H}} g_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v}) = \mathrm{E}|\mathbf{u}(t)|^4, \tag{3.52}$$

since $\{\int_{\mathbb{H}} g_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v})\}_{n=1}^{\infty}$ is a nonnegative sequence of measurable functions. Applying Fatou's lemma for the left-hand side of (3.52),

$$\begin{aligned}\mathrm{E}\int_{\mathbb{H}} |\mathbf{v}|^4\Pi_t(., \mathrm{d}\mathbf{v}) &= \mathrm{E}\int_{\mathbb{H}} \liminf_{n\to\infty} g_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v}) \\ &\leq \liminf_{n\to\infty} \mathrm{E}\int_{\mathbb{H}} g_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v}) = \mathrm{E}|\mathbf{u}(t)|^4.\end{aligned}$$

Taking the supremum on both sides, and applying Theorem 2.3, we have

$$\sup_{0\leq t\leq T} \mathrm{E}\int_{\mathbb{H}} |\mathbf{v}|^4\Pi_t(., \mathrm{d}\mathbf{v}) \leq \mathrm{E}\sup_{0\leq t\leq T} |\mathbf{u}(t)|^4 < \infty. \tag{3.53}$$

Denote $\bar{g}_n(\mathbf{v}) = (|\mathbf{v}|^2\|\mathbf{v}\|^2) \wedge n$ for $n \in \mathbb{N}$. Now consider,

$$\begin{aligned}\mathrm{E}\int_0^T \int_{\mathbb{V}} \bar{g}_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v})\mathrm{d}t &= \mathrm{E}\int_{\mathbb{V}}\int_0^T \bar{g}_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v})\mathrm{d}t \\ &\leq \mathrm{E}\int_{\mathbb{H}}\int_0^T \bar{g}_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v})\mathrm{d}t \\ &= \mathrm{E}[\int_0^T \bar{g}_n(\mathbf{u}(t))\mathrm{d}t]\end{aligned} \tag{3.54}$$

The above result (3.54) holds since $\bar{g}_n(\mathbf{v})$ is a nonnegative bounded Borel function, by Fubini's theorem and result (3.50). By following similar a arguments as in deriving (3.54), we can show that

$$\begin{aligned}\mathrm{E}\int_0^T \int_{\mathbb{V}} |\mathbf{v}|^2\|\mathbf{v}\|^2\Pi_t(., \mathrm{d}\mathbf{v})\mathrm{d}t &= \mathrm{E}\int_0^T \int_{\mathbb{V}} \liminf_{n\to\infty} \bar{g}_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v})\mathrm{d}t \\ &\leq \liminf_{n\to\infty} \mathrm{E}\int_0^T \int_{\mathbb{V}} \bar{g}_n(\mathbf{v})\Pi_t(., \mathrm{d}\mathbf{v})\mathrm{d}t = \mathrm{E}\int_0^T |\mathbf{u}(t)|^2\|\mathbf{u}(t)\|^2\mathrm{d}t.\end{aligned}$$

Then, from Theorem 2.3,

$$\mathrm{E}\int_0^T \int_{\mathbb{V}} |\mathbf{v}|^2\|\mathbf{v}\|^2\Pi_t(., \mathrm{d}\mathbf{v})\mathrm{d}t < \infty. \tag{3.55}$$

Results (3.53) and (3.55) immediately give us the condition (2) of the Definition 3.2. Our main work of the existence of measure-valued solution is verifying the condition

(3) of the Definition 3.2. To obtain the desired result (3.45), we will simply substitute (3.50) to (3.7), but one can find that without further restrictions on $f \in \mathrm{B}_b(\mathbb{H})$, the term $\int_0^t \mathrm{E}(\mathscr{A}f \mid \mathscr{F}_s^Z)\mathrm{d}s$ associated with (3.7) is not well defined and (3.45) is not implied from (3.50) and (3.7). In order to make sense of the terms involve with (3.7), we have to restrict function f to the class of cylindrical test functions ($\mathbb{C}_{\text{cyl}}$). In Theorem 3.5, results (3.13), (3.14), (3.16), (3.19) imply that the term $\int_0^t \mathrm{E}(\mathscr{A}f \mid \mathscr{F}_s^Z)\mathrm{d}s$ is well defined for $f \in \mathbb{C}_{\text{cyl}}$. Result (3.10) in Theorem 3.5 shows that the last term of (3.7) is a martingale. Therefore, we can conclude that all terms associated with (3.7) are well defined when we restrict function f to the class of cylindrical test functions. Then by substituting (3.50) to (3.7), we can obtain the desired result (3.45). □

3.2.2. Uniqueness of Measure-valued Solutions. Let us denote by $\mathrm{S}_t (t \geq 0)$ the Feller semigroup associated with the transition probabilities $\mathrm{P}_t(x, F) = \mathrm{E}(1_F(\mathbf{u}(t)) \mid \mathscr{F}_0)$, ($F \in \mathscr{B}(\mathbb{H})$, where $\mathscr{B}(\mathbb{H})$ consists of all Borel subsets of $\mathbb{H}$):

$$\mathrm{S}_t f(x) = \int_{\mathbb{H}} f(y)\mathrm{P}_t(x, \mathrm{d}y). \tag{3.56}$$

S_t maps $\mathbb{C}_b(\mathbb{H})$ into itself for all $(t \geq 0)$. Let $f \in \mathbb{C}_{\text{cyl}} \subseteq \mathrm{D}(\mathscr{A})$ and consider

$$\mathrm{E}\,|\Pi_t(f) - \Pi_{t'}(f)| = \mathrm{E}\left|\int_{t'}^t \Pi_s(\mathscr{A}f)\mathrm{d}s + \int_{t'}^t [\Pi_s(fh) - \Pi_s(f)\Pi_s(h)] \cdot \mathrm{d}\boldsymbol{\nu}(s)\right| < \infty. \tag{3.57}$$

The above result can be established by using Condition (2) in Definition 3.3, Burkholder's inequality and Cauchy-Schwartz inequality. Then,

$$\lim_{t \to t'} \mathrm{E}\,|\Pi_t(f) - \Pi_{t'}(f)| = 0. \tag{3.58}$$

This gives us

$$\begin{aligned} &\mathrm{E}\,|\Pi_t(f) - \Pi_{t'}(f)| \\ &\geq \int_{\{\omega:|\Pi_t(f)(\omega) - \Pi_{t'}(f)(\omega)| \geq \epsilon\}} |\Pi_t(f)(\omega) - \Pi_{t'}(f)(\omega)|\,\mathrm{P}(\mathrm{d}\omega) \\ &\geq \epsilon \mathrm{P}\{\omega : |\Pi_t(f)(\omega) - \Pi_{t'}(f)(\omega)| \geq \epsilon\}, \quad \text{for all } \epsilon > 0. \end{aligned} \tag{3.59}$$

Combining (3.58) and (3.59), we have

$$\lim_{t \to t'} \mathrm{P}\{\omega : |\Pi_t(f)(\omega) - \Pi_{t'}(f)(\omega)| \geq \epsilon\} = 0, \quad \text{for all } \epsilon > 0. \tag{3.60}$$

Then we can conclude that for each $f \in \mathbb{C}_{\text{cyl}} \subseteq \mathrm{D}(\mathscr{A})$, $\Pi_t(f)$ has a sample continuous path since the result (3.60) and (S_t) is a Feller semigroup and $\mathbb{C}_{\text{cyl}}$ is dense in $\mathbb{C}_b(\mathbb{H})$ (see [17, Section 11.5] and [7, Theorem 2.8]).

Theorem 3.10. *For all $f \in \mathbb{C}_b(\mathbb{H})$, $\Pi_t(f)$ satisfies the equation*

$$\Pi_t(f) = \Pi_0(\mathrm{S}_t f) + \int_0^t [\Pi_s((\mathrm{S}_{t-s}f)h) - \Pi_s(\mathrm{S}_{t-s}f)\Pi_s(h)] \cdot \mathrm{d}\boldsymbol{\nu}(s), \tag{3.61}$$

where $\mathrm{S}_{t-s}f(\mathbf{u}(s)) = \mathrm{E}[f(\mathbf{u}(t)) \mid \mathscr{F}_s]$.

Proof. See [17, Theorem 11.5.1] □

Now let $\{\boldsymbol{v}(t), t \geq 0\}$ be a Wiener process defined on $(\Omega, \mathcal{G}_t, \mathrm{P})$ and take $\Pi_0(f) = \mathrm{E}[f(\mathbf{u}(0))]$ (i.e., $\Pi_0 \in \mathcal{P}(\mathbb{H})$).

We say that $\mathcal{P}(\mathbb{H})$-valued stochastic process $\Pi_t(.)$ is a solution of the equation

$$\Pi_t(f) = \Pi_0(\mathrm{S}_t f) + \int_0^t [\Pi_s((\mathrm{S}_{t-s}f)h) - \Pi_s(\mathrm{S}_{t-s}f)\Pi_s(h)] \cdot \mathrm{d}\boldsymbol{v}(s) \tag{3.62}$$

if it satisfies Conditions (1) and (2) in the Definition 3.3 and for every $t(\geq 0)$, $\Pi_t(f)$ satisfies (3.62) a.s.

Theorem 3.11. *Let* $\{\mathbf{u}(t), t \geq 0\}$ *be the Feller-Markov process which is a solution of the Itô-Lévy stochastic two-dimensional Navier-Stokes equation given by* (2.14). *The observation process* $\mathrm{Z}(t)$ *associate with the signal process* $\mathbf{u}(t)$ *is given by* (3.1). *Assume that the conditions* C1–C3 *hold. Let* $f \in \mathbb{C}_{\mathrm{cyl}} \subseteq \mathrm{D}(\mathcal{A})$. *Then Equation* (3.62) *has a unique solution in the probability measure space* $\mathcal{P}(\mathbb{H})$ *within the class of* $\{\Pi_t \in \mathcal{P}(\mathbb{H}) : \mathrm{E}\int_{\mathbb{H}} |h(\mathbf{v})|^2 \Pi_t(\mathrm{d}\mathbf{v}) \leq C\mathrm{E}|h(\mathbf{u}(t))|^2, \textit{for all } t \in [0, T]\}$:

Proof. We suppose that Π_t and Π_t' are two solutions of (3.62) with the same initial conditions. Our goal is to show that $\Delta_t(f) = \mathrm{E}(|\Pi_t(f) - \Pi_t'(f)|^2) = 0$ for all $t(\geq 0)$ and each $f \in \mathbb{C}_{\mathrm{cyl}}$.
Since two measures Π_t and Π_t' satisfy Condition (2) in Definition 3.3 and $f \in \mathbb{C}_{\mathrm{cyl}}$, we have following estimates:

$$\mathrm{E}|\Pi_t(f)|^2 \leq C\mathrm{E}|f(\mathbf{u}(t))|^2 < \infty, \tag{3.63}$$

$$\mathrm{E}|\acute{\Pi}_t(f)|^2 \leq \acute{C}\mathrm{E}|f(\mathbf{u}(t))|^2 < \infty. \tag{3.64}$$

Using the Triangle inequality with (3.63) and (3.64), we have

$$\Delta_t(f) = \mathrm{E}\left(\left|\Pi_t(f) - \acute{\Pi}_t(f)\right|^2\right) \leq 4\overline{C}\mathrm{E}|f(\mathbf{u}(t))|^2 < \infty, \tag{3.65}$$

where $\overline{C} = \max\{C, \acute{C}\}$.
Let us introduce stopping times involve with the terms $h(\mathbf{u}(t))$ and $\Pi_t(h)$.

$$\tau_1^N = \inf\{t \leq T : |h(\mathbf{u}(t))| \geq N\},$$
$$\tau_2^N = \inf\{t \leq T : |\Pi_t(h)| \geq N\}.$$

Take $\tau_N = \tau_1^N \wedge \tau_2^N$.

Now we will show that $\tau_N \to t$ as $N \to \infty$ for any $t \leq T$. Applying a priori estimates with condition C2, we get,

$$[\mathrm{E}|\Pi_t(h)|]^2 \leq \mathrm{E}|\Pi_t(h)|^2 \leq C\mathrm{E}|h(\mathbf{u}(t))|^2 \leq Cc_h^2\mathrm{E}(1 + |\mathbf{u}(t)|^4) < \infty. \tag{3.66}$$

This gives us that if we define

$$\hat{\Omega}_N := \{\omega \in \Omega : |\Pi_t(h)| < N\}, \tag{3.67}$$

then we have

$$\int_{\hat{\Omega}_N} |\Pi_t(h)|\mathrm{P}(\mathrm{d}\omega) + \int_{\Omega\backslash\hat{\Omega}_N} |\Pi_t(h)|\mathrm{P}(\mathrm{d}\omega) \le \hat{C}. \tag{3.68}$$

Hence drop the first integral and note that $|\Pi_t(h)| \ge N$ in $\Omega\backslash\hat{\Omega}_N$. Then we have

$$\mathrm{P}\left\{\Omega\backslash\hat{\Omega}_N\right\} \le \frac{\hat{C}}{N}. \tag{3.69}$$

Note also that

$$\mathrm{P}\left\{\omega \in \Omega : \tau_1^N < t\right\} = \mathrm{P}\left\{\Omega\backslash\hat{\Omega}_N\right\} \le \frac{\hat{C}}{N}, \quad \text{for any } t \le T. \tag{3.70}$$

Hence, $\limsup_{N\to\infty} \mathrm{P}\{\omega \in \Omega : \tau_1^N < t\} = 0$. Therefore, $\tau_1^N \to t$ as $N \to \infty$. Similarly, we can show that $\tau_2^N \to t$ as $N \to \infty$. Thus, $\tau_N \to t$ as $N \to \infty$ for any $t \le T$. Consider

$$\begin{aligned}
\Delta_{t\wedge\tau_N}(f) &= \mathrm{E}\bigg|\Pi_0(\mathrm{S}_{t\wedge\tau_N}f) + \int_0^{t\wedge\tau_N} \big[\Pi_s((\mathrm{S}_{t\wedge\tau_N-s}f)h) \\
&\quad - \Pi_s(\mathrm{S}_{t\wedge\tau_N-s}f)\Pi_s(h)\big]\cdot \mathrm{d}\boldsymbol{v}(s) - \acute{\Pi}_0(\mathrm{S}_{t\wedge\tau_N}f) \\
&\quad - \int_0^{t\wedge\tau_N} \big[\acute{\Pi}_s((\mathrm{S}_{t\wedge\tau_N-s}f)h) - \acute{\Pi}_s(\mathrm{S}_{t\wedge\tau_N-s}f)\acute{\Pi}_s(h)\big]\cdot \mathrm{d}\boldsymbol{v}(s)\bigg|^2 \\
&= \mathrm{E}\bigg|\int_0^{t\wedge\tau_N} \big[\Pi_s((\mathrm{S}_{t\wedge\tau_N-s}f)h) - \acute{\Pi}_s((\mathrm{S}_{t\wedge\tau_N-s}f)h)\big]\cdot \mathrm{d}\boldsymbol{v}(s) \\
&\quad + \int_0^{t\wedge\tau_N} \big[\Pi_s(\mathrm{S}_{t\wedge\tau_N-s}f)(\acute{\Pi}_s(h) - \Pi_s(h))\big]\cdot \mathrm{d}\boldsymbol{v}(s) \\
&\quad + \int_0^{t\wedge\tau_N} \big[\acute{\Pi}_s(h)(\acute{\Pi}_s(\mathrm{S}_{t\wedge\tau_N-s}f) - \Pi_s(\mathrm{S}_{t\wedge\tau_N-s}f))\big]\cdot \mathrm{d}\boldsymbol{v}(s)\bigg|^2 \\
&\le 3\mathrm{E}\bigg\{\bigg|\int_0^{t\wedge\tau_N} \big[\Pi_s((\mathrm{S}_{t\wedge\tau_N-s}f)h) - \acute{\Pi}_s((\mathrm{S}_{t\wedge\tau_N-s}f)h)\big]\cdot \mathrm{d}\boldsymbol{v}(s)\bigg|^2 \\
&\quad + \bigg|\int_0^{t\wedge\tau_N} \big[\Pi_s(\mathrm{S}_{t\wedge\tau_N-s}f)(\acute{\Pi}_s(h) - \Pi_s(h))\big]\cdot \mathrm{d}\boldsymbol{v}(s)\bigg|^2 \\
&\quad + \bigg|\int_0^{t\wedge\tau_N} \big[\acute{\Pi}_s(h)(\acute{\Pi}_s(\mathrm{S}_{t\wedge\tau_N-s}f) - \Pi_s(\mathrm{S}_{t\wedge\tau_N-s}f))\big]\cdot \mathrm{d}\boldsymbol{v}(s)\bigg|^2\bigg\} \\
&= 3\mathrm{E}\bigg\{\int_0^{t\wedge\tau_N} \big|\Pi_s((\mathrm{S}_{t\wedge\tau_N-s}f)h) - \acute{\Pi}_s((\mathrm{S}_{t\wedge\tau_N-s}f)h)\big|^2 \mathrm{d}s \\
&\quad + \int_0^{t\wedge\tau_N} \big|\Pi_s(\mathrm{S}_{t\wedge\tau_N-s}f)(\acute{\Pi}_s(h) - \Pi_s(h))\big|^2 \mathrm{d}s \\
&\quad + \int_0^{t\wedge\tau_N} \big|\acute{\Pi}_s(h)(\acute{\Pi}_s(\mathrm{S}_{t\wedge\tau_N-s}f) - \Pi_s(\mathrm{S}_{t\wedge\tau_N-s}f))\big|^2 \mathrm{d}s\bigg\}
\end{aligned}$$

$$\leq 3\int_0^{t\wedge\tau_N}\Big\{\Delta_s((\mathrm{S}_{t\wedge\tau_N-s}f)h)+N^2\Delta_s(\mathrm{S}_{t\wedge\tau_N-s}f)$$
$$+|f|^2\Delta_s(h)\Big\}\mathrm{d}s. \tag{3.71}$$

The last inequality of (3.71) follows from Itô isometry and definition of $\Delta_t(f)$.

Now we estimate the terms involve in right-hand side of (3.71) in their order of appearance. Using the fact that $h(\mathbf{u}(t))$ is measurable with respect to $\mathscr{F}_t^Z$, applying Jensen's inequality with condition C2,

$$\begin{aligned}\Delta_s((\mathrm{S}_{t\wedge\tau_N-s}f)h) &\leq 4C_1\mathrm{E}|h(\mathbf{u}(s))\mathrm{S}_{t\wedge\tau_N-s}f(\mathbf{u}(s))|^2\\ &\leq 4C_1\mathrm{E}\left[\mathrm{E}\left(|h(\mathbf{u}(s))f(\mathbf{u}(t\wedge\tau_N))|^2|\mathscr{F}_s^Z\right)\right]\\ &\leq 4C_1\mathrm{E}\left[|h(\mathbf{u}(s))|^2|f(\mathbf{u}(t\wedge\tau_N))|^2\right],\quad \text{for } 0\leq s\leq t\leq T, \end{aligned}\tag{3.72}$$

$$\Delta_s(h)\leq 4C_2\mathrm{E}|h(\mathbf{u}(s))|^2,\quad \text{for } 0\leq s\leq t\leq T \tag{3.73}$$

and

$$\Delta_s(\mathrm{S}_{t\wedge\tau_N-s}f)\leq 4C_3\mathrm{E}|f(\mathbf{u}(t\wedge\tau_N))|^2,\quad \text{for } 0\leq t\leq T. \tag{3.74}$$

From (3.71), (3.72), (3.73), and (3.74), we obtain

$$\begin{aligned}\Delta_{t\wedge\tau_N}(f) &\leq 4\cdot 3\cdot C_m\Big\{\int_0^{t\wedge\tau_N}\mathrm{E}\left[|h(\mathbf{u}(s))|^2|f(\mathbf{u}(t\wedge\tau_N))|^2\right]\mathrm{d}s\\ &\quad+N^2\int_0^{t\wedge\tau_N}\mathrm{E}|f(\mathbf{u}(t))|^2\mathrm{d}s+|f|^2\int_0^{t\wedge\tau_N}\mathrm{E}|h(\mathbf{u}(s))|^2\mathrm{d}s\Big\}\\ &\leq 4\cdot 3^2C_mN^2|f|^2(t\wedge\tau_N). \end{aligned}\tag{3.75}$$

where $C_m=\max\{C_1,C_2,C_3\}$.

Substituting the estimate (3.75) into the right-hand side of (3.71),

$$\begin{aligned}\Delta_{t\wedge\tau_N}(f) &\leq 4\cdot 3^3\cdot C_mN^2\Big\{\mathrm{E}\int_0^{t\wedge\tau_N}|(\mathrm{S}_{t\wedge\tau_N-s}f)h|^2s\mathrm{d}s+\mathrm{E}\int_0^{t\wedge\tau_N}|h(\mathbf{u}(s))|^2s\mathrm{d}s\\ &\quad+\mathrm{E}\int_0^{t\wedge\tau_N}|\mathrm{S}_{t\wedge\tau_N-s}f|^2s\mathrm{d}s\Big\}\\ &\leq 4\cdot 3^3\cdot C_mN^4|f|^2\frac{(t\wedge\tau_N)^2}{2}. \end{aligned}\tag{3.76}$$

We can get the following estimate by repeating the above procedure n times:

$$\Delta_{t\wedge\tau_N}(f)\leq 4\cdot 3^{n+1}\cdot C_m|f|^2N^{2n}\frac{(t\wedge\tau_N)^n}{n!}<\infty. \tag{3.77}$$

By letting $n\to\infty$ of the right-hand side of (3.77) for fixed N, we can get the uniqueness result for measure-valued solutions of (3.62) up to stopping time τ_N.

Now let $\overline{\Omega} = \{\omega \in \Omega : \Delta_{t \wedge \tau_N}(f) = 0\}$. Then we have $\Delta_t(f) = 0$ for $t > \tau_N$, $0 \le t \le T$ on the set $\Omega_N = \{\omega \in \Omega : \tau_N = T\} \cap \overline{\Omega}$, since $\Delta_t(f) = 0$ may not hold for $t > \tau_N$. Since we have $P(\Omega_N) \to 1$ as $N \to \infty$, we will get $\Delta_t(f) = 0$ for $\omega \in \cup_{N=1}^{\infty} \Omega_N$ with $P(\cup_{N=1}^{\infty} \Omega_N) = 1$. This implies that uniqueness result holds for whole interval $[0, T]$. □

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