Water Waves

Wilton Ripples in Weakly Nonlinear Models of Water Waves: Existence and Computation

--Manuscript Draft--

Manuscript Number:	
Full Title:	Wilton Ripples in Weakly Nonlinear Models of Water Waves: Existence and Computation
Article Type:	Original Article
Funding Information:	National Science Foundation (DMS-1813033) — Dr. David Nicholls Air Force Office of Scientific Research — Dr. Benjamin Akers Office of Naval Research — Dr. Benjamin Akers
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Water Waves manuscript No.
(will be inserted by the editor)

Wilton Ripples in Weakly Nonlinear Models of Water Waves: Existence and Computation

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Received: date / Accepted: date

Abstract In this contribution we prove that small amplitude, resonant harmonic traveling waves (Wilton Ripples) exist in a family of weakly nonlinear PDEs which model water waves. The proof is inspired by that of Reeder & Shinbrot (1981) and complements the authors' recent, independent result proven by a perturbative technique (Akers & Nicholls, 2020). The method is based on a Banach Fixed Point Iteration and, in addition to proving that this iteration has Wilton Ripples as a fixed point, we use it as a numerical method for simulating these solutions. The output of this numerical scheme and its performance are evaluated against a Quasi–Newton iteration.

Keywords Wilton Ripples · weakly nonlinear PDEs · Whitham equation · Benjamin equation · Kawahara equation · Akers–Milewski equation.

1 Introduction

In this work we consider the existence of a family of resonant traveling waves, first described by Wilton [61], in a class of weakly nonlinear wave equations which model water waves. These waves occur in the models at certain *critical*

B.F.A. was supported during the preparation of this manuscript by the Air Force Office of Sponsored Research and the Office of Naval Research. D.P.N. gratefully acknowledges support from the National Science Foundation through grant No. DMS–1813033.

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values of the surface tension [42]. The surface tension coefficient appears as a parameter in these PDEs and, for special values of this parameter, the operator defining the linear portion of the equation has a two dimensional kernel implying that the linear problem supports *two* co–propagating resonant harmonics. Continuous branches of these near–bichromatic waves exist about special combinations of the harmonics which are called "Wilton Ripples." These waves have been studied for over a century now including asymptotic [61,46,34,39], numerical [11,60,19,14], and experimental work [45,35,50]. In this paper we present a rigorous treatment of the phenomena in a class of weakly nonlinear PDEs, proving existence of these resonant ripples and building a rapid and robust numerical method based directly upon the iteration we describe in our proof. We point out that, while connected to our previous work on this topic [7], our iteration does not deliver the analyticity of solution branches required of the previous numerical method of the first author [11].

Our method of proof is to apply a Banach Fixed Point Iteration (FPI) [30], however, suitable "initiation" is required to demonstrate the existence of a fixed point. In more detail, in the case of Stokes waves (which feature only a one–dimensional null space in the linear operator) the first step must be carefully described. By contrast, in the case of Wilton Ripples the first *two* steps must be constructed in order to assure that the correct type of solution is found.

Away from special surface tension values, small amplitude traveling solutions are monochromatic so that the linear problem has a one–dimensional solution space, resulting in Stokes waves. These waves have a significant history of both asymptotic [56,33,23,44] and numerical study [52,53,48,3]. Their rigorous study is also well developed as Stokes waves are known to exist [43, 57], and be parametrically analytic in amplitude [21,22,49]. The global bifurcation problem has been described and extreme waves on branches have been characterized [13].

Wilton Ripples have a shorter and less complete history of rigorous study. Often the resonant and non–resonant cases are studied separately due to the change in form of the wave's asymptotics. The bifurcation structure has been described for the non–resonant case for both potential flow [10] and the hydroelastic water wave problem [58]. Reeder and Shinbrot discuss the existence and analyticity of small resonant Wilton Ripples in potential flow [51] while the bifurcation structure of Wilton Ripples has recently been described for the Whitham equation [29]. Although our motivation stems from water waves, here we consider Wilton Ripples in a family of weakly nonlinear models, including the Kawaraha [40], Benjamin [15], Whitham [47] and Akers–Milewski [2] equations. In a recent paper the authors used a perturbation theoretic method to demonstrate that resonant triad ripples exist along analytic bifurcation curves in this class of model equations [7]. Although the proof in [7] gives a stronger result, we believe that the proof in this work is also of value, both for its associated numerical method and its potential to be applied to the stability problem (perhaps to amplitudes at which the spectrum is continuously, but not analytically, connected to the flat configuration).

As a bifurcation problem, the computation of traveling waves shares similar structure to that of the computation of their spectral data. This analogy forms the inspiration for the numerical methods in [4–6]. In the setting of this analogy, resonant traveling waves correspond to eigenvalue collisions; thus, while resonant traveling waves might be considered "exotic" and "uncommon," eigenvalue collisions are viewed as the *primary* mechanism for instability and are therefore the most interesting and important parts of the spectrum [9]. In the future, we hope to develop this proof and numerical method for traveling waves to complement traditional studies of the spectrum, both numerical [28,27,59] and theoretical [17,36–38,8]. Recently Creedon, Deconinck and Trichtchenko combined the Fourier–Floquet–Hill method from [28] with the asymptotic ideas in [8] to compute high frequency spectrum in a Boussinesq–Whitham model [25] and the Kawahara equation [26]. Applying the ideas of this paper to the spectral data would be a natural complement to the work this group is doing on high–frequency instabilities.

In this paper we consider a family of model equations whose linear phase speed allows two co–propagating resonant harmonics. In these equations we show that small amplitude Wilton Ripples exist, and that there are branches of these waves, at least up to some maximal amplitude. The paper is organized as follows: In § 2 we state the weakly nonlinear wave equations which govern the physical phenomena of interest in this paper. We give a proof of the existence of Stokes waves in § 3, with the study of the linear problem § 3.1, a description of the iteration in § 3.2, and the result in § 3.3. Subsequent to this, we establish the existence of Wilton Ripples in § 4, with the study of the linear and quadratic problems in § 4.1 & § 4.2, a description of the iteration in § 4.3, and the result in § 4.4. We close with numerical results in § 5 and concluding remarks in § 6.

2 Governing Equations

In this paper we study x–periodic solutions of weakly nonlinear dispersive wave equations of the form

$$\partial_t u + \mathcal{L}\partial_x u - N(u) = 0, \quad u(x+2\pi, t) = u(x,t), \tag{1}$$

where $\mathcal{L}$ is a linear operator (see Table 1 for examples considered here), which model the full water wave equations [42] in the limit of long wavelength and weak nonlinearity. Among the myriad choices, we focus on those which permit co–propagating traveling waves in the linear regime. Examples are the Whitham [47], Akers–Milewski [2], Benjamin [15], and Kawahara [40] equations (see Table 1). All of these share a common nonlinearity (after a suitable rescaling) which we take to be

$$N(u) = \frac{1}{2}\partial_x \left[u^2\right] = u\partial_x u.$$

We now turn to traveling wave solutions of (1)

$$u(x,t) = f(x - ct)$$

which satisfy

$$-c\partial_x f + \mathcal{L}\partial_x f - \frac{1}{2}\partial_x \left[f^2\right] = 0.$$

After integrating once with respect to x we find

$$(-c + \mathcal{L})f = -\frac{1}{2}f^2.$$

which we write as

$$(-c_0 + \mathcal{L})f = (c - c_0)f - \frac{1}{2}f^2,$$

where we define c_0 presently. By defining

$$\mathcal{R} := -c_0 + \mathcal{L},$$

we express this as

$$\mathcal{R}f = \left((c - c_0) - \frac{1}{2}f\right) f. \tag{2}$$

3 Stokes Waves

In order to give some notation and build intuition for the Wilton Ripple case, we begin with the simpler case of the classical Stokes waves which result from simple bifurcation [24] in (2). For this we note that there is a speed c_0 such that the operators $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ have one–dimensional null spaces spanned by ϕ and ψ, respectively, so that

$$\mathcal{R}\phi = 0, \quad \mathcal{R}^*\psi = 0, \quad \phi(x) = \psi(x) = \frac{\cos(x)}{\sqrt{\pi}},$$

and we note that

$$\langle \phi, \phi \rangle = \langle \psi, \psi \rangle = \langle \phi, \psi \rangle = 1.$$

3.1 Preparing the Iteration: Linear Order

We now seek a solution of (2) of the form

$$f = \varepsilon f_1 + \varepsilon^2 F, \quad c = c_0 + \varepsilon s. \tag{3}$$

Inserting these into (2) gives

$$\mathcal{R}[\varepsilon f_1 + \varepsilon^2 F] = \left(\varepsilon s - \frac{1}{2}\varepsilon f_1 - \frac{1}{2}\varepsilon^2 F\right)\left(\varepsilon f_1 + \varepsilon^2 F\right). \tag{4}$$

At linear order we find

$$\mathcal{R}f_1 = 0$$

which demands that $f_1 \in \mathcal{N}(\mathcal{R})$ so that $f_1 = \alpha\phi$; without loss of generality we may select $\alpha = 1$.

3.2 The Stokes Iteration

Using this choice of f_1 we return to (4) and, using the properties of f_1, and dividing by ε^2 we find

$$\mathcal{R}F - sf_1 = -\frac{1}{2}f_1^2 + \varepsilon\left\{sF - f_1 F - \frac{1}{2}\varepsilon F^2\right\}.$$

Inspired by this we conduct the following Stokes iteration

$$\mathcal{R}F_n - s_n f_1 = -\frac{1}{2}f_1^2 + \varepsilon G(F_{n-1}, s_{n-1}; \varepsilon), \tag{5}$$

where

$$G(F, s; \varepsilon) := sF - f_1 F - \frac{1}{2}\varepsilon F^2.$$

Our procedure for solving (5) at each iteration is two–step:

1. First, we solve for the speed correction s_n by projecting the right–hand-side of (5) and $s_n f_1$ onto the range of $\mathcal{R}$. This is enforced by demanding that this be orthogonal to the null space of its adjoint,

$$\begin{aligned} 0 &= \left\langle s_n f_1 - \frac{1}{2}f_1^2 + \varepsilon G(F_{n-1}, s_{n-1}; \varepsilon), \psi \right\rangle \\ &= s_n \langle \phi, \psi \rangle - \frac{1}{2}\left\langle \phi^2, \psi \right\rangle + \varepsilon \langle G(F_{n-1}, s_{n-1}; \varepsilon), \psi \rangle . \end{aligned}$$

   Using $\langle \phi, \psi \rangle = 1$ and (from our choices of $\{\phi, \psi\}$) $\left\langle \phi^2, \psi \right\rangle = 0$ we find that

$$s_n = -\varepsilon \langle G(F_{n-1}, s_{n-1}; \varepsilon), \psi \rangle .$$

2. With this choice of s_n we know that (5) is uniquely solvable provided that we demand orthogonality of the solution to the null space of $\mathcal{R}$. So, we can write

$$\begin{aligned} &F_n = \mathcal{R}^{-1}\left[s_n f_1 - \frac{1}{2}f_1^2 + \varepsilon G(F_{n-1}, s_n; \varepsilon)\right], \\ &\langle \phi, F_n \rangle = 0. \end{aligned}$$

   This procedure can be replicated for the linearized version of this problem

$$\mathcal{R}F - sf_1 = -\frac{1}{2}f_1^2 + J,$$

and rigorous estimates derived for the unique solution. This is summarized in the following “elliptic estimate” [7].

Theorem 1 *For any real $\sigma \geq 0$, given $J \in H^\sigma$ there exists a unique solution pair (F, s) of*

$$\mathcal{R}F - sf_1 = -\frac{1}{2}f_1^2 + J,$$
$$\langle \phi, F \rangle = 0,$$

such that $F \in H^{\sigma+\tau}$ and

$$|s| \leq C_e \|J\|_{H^\sigma},$$
$$\|F\|_{H^{\sigma+\tau}} \leq C_e \left\| sf_1 - \frac{1}{2}f_1^2 + J \right\|_{H^\sigma}$$

for some $C_e > 0$ and

$$\tau = \begin{cases} 1/2, & \text{Whitham (A)}, \\ 1, & \text{Akers–Milewski (B)}, \\ 2, & \text{Benjamin (C) and Hyper–Akers–Milewski (D)}, \\ 4, & \text{Kawahara (E) and Hyper–Whitham (F)}. \end{cases}$$

3.3 Banach Fixed Point Formulation

We now write this as an FPI so that we can appeal to the Banach Fixed Point Theorem [30]. For this we write (2) as

$$\mathcal{R}F - f_1 s = -\frac{1}{2}f_1^2 + \varepsilon G(F, s; \varepsilon),$$
$$\langle \phi, F \rangle = 0,$$

or

$$M \begin{pmatrix} F \\ s \end{pmatrix} = \begin{pmatrix} -(1/2)f_1^2 + \varepsilon G(F, s; \varepsilon) \\ 0 \end{pmatrix}, \quad M := \begin{pmatrix} \mathcal{R} & -f_1 \\ \langle \phi, \cdot \rangle & 0 \end{pmatrix}.$$

We now express this as $x = T(x)$ where

$$x := \begin{pmatrix} F \\ s \end{pmatrix}, \quad T(x) := M^{-1} \begin{pmatrix} -(1/2)f_1^2 + \varepsilon G(F, s; \varepsilon) \\ 0 \end{pmatrix},$$

and seek a fixed point of the iteration

$$x_n = T(x_{n-1}).$$

From Theorem 1 we know that M^{-1} is well–defined and that the action of $x^* = T(x)$ can be given by

$$s^* = -\varepsilon \langle G(F, s; \varepsilon), \psi \rangle, \tag{6a}$$

$$F^* = \mathcal{R}^{-1} \left[s^* f_1 - \frac{1}{2}f_1^2 + \varepsilon G(F, s; \varepsilon) \right]. \tag{6b}$$

A fixed point of the iteration $x_n = T(x_{n-1})$ would solve $x = T(x)$ and represent a solution of (2). To achieve this we appeal to Banach's Fixed Point Theorem [30] which requires a Banach space,

$$X = H^{\sigma+\tau} \times \mathbf{R},$$

and a distance,

$$d(x,y)^2 = \|F_x - F_y\|^2_{H^{\sigma+\tau}} + |s_x - s_y|^2 .$$

To show that T is a contraction, we need the following classical Lemma [41,1,31,30].

Lemma 1 *Provided that $\sigma > 1/2$, if $f, g \in H^\sigma$ then $fg \in H^\sigma$ and*

$$\|fg\|_{H^\sigma} \leq M \|f\|_{H^\sigma} \|g\|_{H^\sigma} ,$$

for some $M > 0$.

With this we can establish the following crucial estimate which demonstrates that T is a contraction for ε sufficiently small.

Lemma 2 *Provided that $\sigma > 1/2$ and $x, y \in B_R(0) \subset X$ we have that*

$$d(T(x), T(y))^2 \leq \varepsilon^2 C(R)^2 d(x,y)^2,$$

where $C(R)$ is a quadratic function of R.

Proof We begin with

$$\begin{aligned} d(T(x), T(y))^2 &= d(x^*, y^*)^2 \\ &= \left\|F_x^* - F_y^*\right\|^2_{H^{\sigma+\tau}} + \left|s_x^* - s_y^*\right|^2 . \end{aligned}$$

The first term can be estimated

$$\begin{aligned} \left\|F_x^* - F_y^*\right\|_{H^{\sigma+\tau}} &= \left\| \mathcal{R}^{-1} \left[s_x^* f_1 - \frac{1}{2} f_1^2 + \varepsilon G(F_x, s_x; \varepsilon) \right] \right. \\ &\qquad \left. - \mathcal{R}^{-1} \left[s_y^* f_1 - \frac{1}{2} f_1^2 + \varepsilon G(F_y, s_y; \varepsilon) \right] \right\|_{H^{\sigma+\tau}} \\ &\leq \left\| (s_x^* - s_y^*)\phi + \varepsilon (\Delta G) \right\|_{H^\sigma} \\ &\leq \left| s_x^* - s_y^* \right| \|\phi\|_{H^\sigma} + \varepsilon \|\Delta G\|_{H^\sigma} , \end{aligned}$$

where we have used Theorem 1, $f_1 = \phi$, and defined

$$\Delta G := G(F_x, s_x; \varepsilon) - G(F_y, s_y; \varepsilon).$$

Using $(a+b)^2 \leq 2(a^2 + b^2)$, we now have

$$d(T(x), T(y))^2 \leq \left(1 + 2 \|\phi\|^2_{H^\sigma}\right) \left|s_x^* - s_y^*\right|^2 + 2\varepsilon^2 \|\Delta G\|^2_{H^\sigma} . \tag{7}$$

Now, the difference in the speed corrections can be estimated from the solution formula (6a) and the Cauchy–Schwartz Inequality

$$\begin{aligned} \left| s_x^* - s_y^* \right| &= \varepsilon \left| \langle G(F_x, s_x; \varepsilon) - G(F_y, s_y; \varepsilon), \psi \rangle \right| \\ &\le \varepsilon \left\| \Delta G \right\|_{H^0} \left\| \psi \right\|_{H^0} \le \varepsilon \left\| \Delta G \right\|_{H^\sigma}, \end{aligned}$$

and we have used $\sigma > 1/2 > 0$ and $\|\psi\|_{H^0} = \|\psi\|_{L^2} = 1$. With this (7) tells us that

$$d(T(x), T(y))^2 \le \varepsilon^2 \left(3 + 2 \left\| \phi \right\|_{H^\sigma}^2 \right) \left\| \Delta G \right\|_{H^\sigma}^2. \tag{8}$$

Finally, we estimate the term ΔG with the following manipulations

$$\begin{aligned} \left\| \Delta G \right\|_{H^\sigma} &\le \left\| s_x F_x - \phi F_x - \frac{1}{2} \varepsilon F_x^2 - \left(s_y F_y - \phi F_y - \frac{1}{2} \varepsilon F_y^2 \right) \right\|_{H^\sigma} \\ &\le \left| s_x - s_y \right| \left\| F_x \right\|_{H^\sigma} + \left| s_y \right| \left\| F_x - F_y \right\|_{H^\sigma} + M \left\| \phi \right\|_{H^\sigma} \left\| F_x - F_y \right\|_{H^\sigma} \\ &\quad + \frac{1}{2} \varepsilon M \left\| F_x + F_y \right\|_{H^\sigma} \left\| F_x - F_y \right\|_{H^\sigma}. \end{aligned}$$

Since $x, y \in B_R(0)$ we have that

$$\max \left\{ s_m, \left\| F_m \right\|_{H^\sigma} \right\} < R, \quad m \in \{x, y\},$$

and since

$$\max \left\{ \left| s_x - s_y \right|, \left\| F_x - F_y \right\|_{H^{\sigma+\tau}} \right\} \le d(x, y),$$

we discover that

$$\begin{aligned} \left\| \Delta G \right\|_{H^\sigma} &\le \left| s_x - s_y \right| R + R \left\| F_x - F_y \right\|_{H^\sigma} + M \left\| \phi \right\|_{H^\sigma} \left\| F_x - F_y \right\|_{H^\sigma} \\ &\quad + \frac{1}{2} \varepsilon M (2R) \left\| F_x - F_y \right\|_{H^\sigma} \\ &\le \left\{ 2R + M \left\| \phi \right\|_{H^\sigma} + \varepsilon M R \right\} d(x, y). \end{aligned}$$

Finally, with this, (8) now gives

$$\begin{aligned} d(T(x), T(y))^2 &\le \varepsilon^2 \left(3 + 2 \left\| \phi \right\|_{H^\sigma}^2 \right) \left\{ 2R + M \left\| \phi \right\|_{H^\sigma} + \varepsilon M R \right\}^2 d(x, y)^2 \\ &\le \varepsilon^2 C(R)^2 d(x, y)^2, \end{aligned}$$

where

$$C(R)^2 := \left(3 + 2 \left\| \phi \right\|_{H^\sigma}^2 \right) \left\{ 2R + M \left\| \phi \right\|_{H^\sigma} + \varepsilon M R \right\}^2,$$

and we are done.

Now, using Lemma 2 we can show that, for sufficiently small ε, if the sequence $x_n = T(x_{n-1})$ starts in $B_R(0)$ it always stays there.

Lemma 3 *Provided that $\sigma > 1/2$, if $x_0 \in B_R(0)$ and $x_n = T(x_{n-1})$ then, for any $R > 0$, there exists $\varepsilon > 0$ such that $x_n \in B_R(0)$ for all $n \ge 0$.*

Proof We work by induction. Given any $x_0 \in B_R(0)$ we have

$$d(x_0, 0) \leq R,$$

and we are done with the base step. Now, suppose that $x_n \in B_R(0)$ for all $n < N$ and consider

$$\begin{aligned} d(x_N, 0) &= d(T(x_{N-1}), 0) \\ &\leq \varepsilon C(R) d(x_{N-1}, 0) \\ &\leq \varepsilon C(R) R \\ &\leq R, \end{aligned}$$

provided that $\varepsilon < 1/C(R)$.

With Lemma 3 in hand we are now able to prove our main result for Stokes waves.

Theorem 2 *Provided that $\sigma > 1/2$, for ε sufficiently small the mapping T is a contraction on $B_R(0)$ implying that, for ε sufficiently small, there exists a unique Stokes wave of the form* (3)*.*

Proof Consider $0 \leq q < 1$ and suppose that $x, y \in B_R(0)$. From Lemma 2 we have that

$$d(T(x), T(y)) \leq \varepsilon C(R) d(x, y) \leq q d(x, y),$$

provided that $\varepsilon < q/C(R)$.

4 Wilton Ripples

We now turn to the case of Wilton Ripples and the situation where there is a speed c_0 such that the operator $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ have two–dimensional null spaces spanned by ϕ_m and ψ_m $(m = 1, 2)$ respectively, so that

$$\mathcal{R}\phi_m = 0, \quad \mathcal{R}^*\psi_m = 0, \quad \phi_m(x) = \psi_m(x) = \frac{\cos(mx)}{\sqrt{\pi}}; \quad m = 1, 2,$$

where we observe that

$$\langle \phi_m, \phi_\ell \rangle = \delta_{m,\ell}, \quad \langle \psi_m, \psi_\ell \rangle = \delta_{m,\ell}, \quad \langle \phi_m, \psi_\ell \rangle = \delta_{m,\ell},$$

and $\delta_{m,\ell}$ is the Kronecker delta function.

4.1 Preparing the Iteration: Linear Order

As before, we seek a solution of (2) of the form

$$f = \varepsilon f_1 + \varepsilon^2 F, \quad c = c_0 + \varepsilon s.$$

Inserting these into (2) gives

$$\mathcal{R}[\varepsilon f_1 + \varepsilon^2 F] = \left(\varepsilon s - \frac{1}{2} \varepsilon f_1 - \frac{1}{2} \varepsilon^2 F \right) \left(\varepsilon f_1 + \varepsilon^2 F \right).$$

At linear order we find

$$\mathcal{R} f_1 = 0$$

which demands that $f_1 \in \mathcal{N}(\mathcal{R})$ so

$$f_1 = \alpha \phi_1 + \beta \phi_2, \quad \alpha, \beta \in \mathbf{R}.$$

As before, we may, without loss of generality, assume that $\alpha = 1$, however, β is still free to be chosen. This must be done at the next order.

4.2 Preparing the Iteration: Quadratic Order

In contrast to the case of Stokes waves, for Wilton Ripples additional initiation is required to *quadratic* order rather than just *linear*. For this, we enhance the form of our solution of (2) to be of the form

$$f = \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 F, \quad c = c_0 + \varepsilon c_1 + \varepsilon^2 s. \tag{9}$$

Inserting these into (2) gives

$$\begin{aligned} &\mathcal{R}[\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 F] \\ &\qquad = \left(\varepsilon c_1 + \varepsilon^2 s - \frac{1}{2} \varepsilon f_1 - \frac{1}{2} \varepsilon^2 f_2 - \frac{1}{2} \varepsilon^3 F \right) \left(\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 F \right). \end{aligned} \tag{10}$$

By our choice of f_1 this equation is satisfied at linear order for *any* choice of β, and we must study the quadratic order equation

$$\mathcal{R} f_2 - c_1 f_1 = -\frac{1}{2} f_1^2.$$

The most general solution of this equation can be written

$$f_2 = a \phi_1 + b \phi_2 + \Phi, \quad \langle \Phi, \phi_1 \rangle = \langle \Phi, \phi_2 \rangle = 0,$$

and we follow the lead of [7] by selecting $a = 0$, and finding the triple $\{\Phi, c_1, \beta\}$ which satisfies

$$\mathcal{R} \Phi - c_1 f_1 = -\frac{1}{2} f_1^2.$$

Following the procedure outlined in [7] we insist that

$$0 = \left\langle c_1 f_1 - \frac{1}{2} f_1^2, \psi_m \right\rangle, \quad m = 1, 2,$$

or, expanding,

$$0 = \left\langle c_1 \phi_1 + c_1 \beta \phi_2 - \frac{1}{2} \phi_1^2 - \beta \phi_1 \phi_2 + \frac{1}{2} \beta^2 \phi_2^2, \psi_m \right\rangle, \quad m = 1, 2.$$

If we define

$$C_{jm} := \langle \phi_j, \psi_m \rangle, \quad C_{j\ell m} := \langle \phi_j \phi_\ell, \psi_m \rangle,$$

we can show (c.f., [7], Lemma 4) that (from our choices of $\{\phi_1, \phi_2, \psi_1, \psi_2\}$)

$$C_{11} = C_{22} = 1, \quad C_{12} = C_{21} = 0, \tag{11a}$$

$$C_{112} = C_{121} = C_{211} = 1/2, \tag{11b}$$

$$C_{111} = C_{221} = C_{212} = C_{122} = C_{222} = 0. \tag{11c}$$

We find that for $m = 1, 2$, respectively,

$$c_1 - (1/2)\beta = 0, \quad c_1 \beta - 1/4 = 0.$$

These can be solved sequentially to give

$$\beta = \pm \frac{1}{\sqrt{2}}, \quad c_1 = \pm \frac{1}{2\sqrt{2}}, \tag{12}$$

and since we have $(c_1 f_1 - (1/2) f_1^2)$ in the range of $\mathcal{R}$ we can solve

$$\Phi = \mathcal{R}^{-1} \left[c_1 f_1 - \frac{1}{2} f_1^2 \right].$$

Before proceeding we note that, while we lose no generality by setting $a = 0$ we cannot yet choose b, and this is one of the unknowns of our iteration. To summarize our developments thus far, we seek solutions of (2) of the form

$$f = \varepsilon(\phi_1 + \beta \phi_2) + \varepsilon^2 (b \phi_2 + \Phi) + \varepsilon^3 F, \quad c = c_0 + \varepsilon c_1 + \varepsilon^2 s, \tag{13}$$

where $\{c_0, \phi_1, \phi_2, \Phi, c_1, \beta\}$ are now known, and we must determine $\{F, s, b\}$.

4.3 The Wilton Iteration

Using these choices we return to (10) which is now satisfied at linear and quadratic orders. Using the fact that $f_1, b\phi_2 \in \mathcal{N}(\mathcal{R})$ and dividing by ε^3 we find

$$\begin{aligned}\mathcal{R}F &= \left\{ \left(c_1 - \frac{1}{2}f_1\right) f_2 + \left(s - \frac{1}{2}f_2\right) f_1 \right\} \\ &\quad + \varepsilon \left\{ \left(c_1 - \frac{1}{2}f_1\right) F + \left(s - \frac{1}{2}f_2\right) f_2 - \frac{1}{2}f_1 F \right\} \\ &\quad + \varepsilon^2 \left\{ \left(s - \frac{1}{2}f_2\right) F - \frac{1}{2}f_2 F \right\} + \varepsilon^3 \left\{ -\frac{1}{2}F^2 \right\} \\ &= \{(c_1 - f_1)f_2 + s f_1\} + \varepsilon G(F, s, b),\end{aligned}$$

where

$$G(F, s, b; \varepsilon) := \left\{ (c_1 - f_1)F + \left(s - \frac{1}{2}f_2\right) f_2 \right\} + \varepsilon \left\{ (s - f_2)\, F \right\} + \varepsilon^2 \left\{ -\frac{1}{2}F^2 \right\}.$$

Using the fact that $f_2 = b\phi_2 + \Phi$ and simpilfying we find

$$\mathcal{R}F - s f_1 - (c_1 - f_1) b\phi_2 = (c_1 - f_1)\Phi + \varepsilon G(F, s, b; \varepsilon).$$

Inspired by this we conduct the following Wilton iteration

$$\mathcal{R}F_n - s_n f_1 - (c_1 - f_1) b_n \phi_2 = (c_1 - f_1)\Phi + \varepsilon G(F_{n-1}, s_{n-1}, b_{n-1}; \varepsilon). \quad (14)$$

Our procedure for solving (14) at each iteration is two–step:

1. First, we solve for the speed corrections $\{s_n, b_n\}$ by projecting the right–hand-side of (14) and $s_n f_1 + (c_1 - f_1) b_n \phi_2$ onto the range of $\mathcal{R}$. This is enforced by demanding that this be orthogonal to the null space of its adjoint,

$$0 = \langle s_n f_1 + (c_1 - f_1) b_n \phi_2 + (c_1 - f_1)\Phi + \varepsilon G(F_{n-1}, s_{n-1}, b_{n-1}; \varepsilon), \psi_m \rangle\,,$$

$m = 1, 2$. Using (11) we find that

$$\begin{pmatrix} 1 & -1/2 \\ \beta & c_1 \end{pmatrix} \begin{pmatrix} s_n \\ b_n \end{pmatrix} = - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},$$

where

$$Q_m := \langle (c_1 - f_1)\Phi + \varepsilon G(F_{n-1}, s_{n-1}, b_{n-1}; \varepsilon), \psi_m \rangle\,, \quad m = 1, 2.$$

Using the fact that $c_1 = \beta/2$, c.f. (12), this can be solved *explicitly* as

$$s_n = -\frac{1}{2}\left(Q_1 + \frac{Q_2}{\beta} \right) \quad (15a)$$

$$b_n = -\frac{1}{2}\left(-2Q_1 + \frac{2Q_2}{\beta} \right). \quad (15b)$$

We also know from (12) that $\beta = \pm 1/\sqrt{2}$ so that once a choice is made, these formulas give the unique solution.

2. With this choice of $\{s_n, b_n\}$ we know that (14) is uniquely solvable provided that we demand orthogonality to the null space of $\mathcal{R}$. So, we can write

$$F_n = \mathcal{R}^{-1}\left[s_n f_1 + (c_1 - f_1) b_n \phi_2 + (c_1 - f_1)\Phi + \varepsilon G(F_{n-1}, s_{n-1}, b_{n-1}; \varepsilon)\right], \tag{16a}$$
$$\langle \phi_1, F_n \rangle = 0 \tag{16b}$$
$$\langle \phi_2, F_n \rangle = 0. \tag{16c}$$

This procedure can be replicated for the linearized version of this problem

$$\mathcal{R}F - s f_1 - (c_1 - f_1) b \phi_2 = (c_1 - f_1)\Phi + J,$$

and rigorous estimates derived for the unique solution. This is summarized in the following "elliptic estimate" [7].

Theorem 3 *For any real $\sigma \geq 0$, given $J \in H^\sigma$ there exists a unique solution triple $\{F, s, b\}$ of*

$$\begin{aligned} &\mathcal{R}F - s f_1 - (c_1 - f_1) b \phi_2 = (c_1 - f_1)\Phi + J, \\ &\langle \phi_1, F \rangle = 0, \\ &\langle \phi_2, F \rangle = 0, \end{aligned}$$

such that $F \in H^{\sigma+\tau}$ and

$$\max\left\{ \|F\|_{H^{\sigma+\tau}}, |s|, |b| \right\} \leq C_e \left\| (c_1 - f_1)\Phi + J \right\|_{H^\sigma}$$

for some $C_e > 0$ and

$$\tau = \begin{cases} 1/2, & \textit{Whitham (A)}, \\ 1, & \textit{Akers–Milewski (B)}, \\ 2, & \textit{Benjamin (C) and Hyper–Akers–Milewski (D)}, \\ 4, & \textit{Kawahara (E) and Hyper–Whitham (F)}. \end{cases}$$

4.4 Banach Fixed Point Formulation

We now write this as an FPI so that we can appeal to the Banach Fixed Point Theorem [30]. To accomplish this we write (2) as

$$\begin{aligned} &\mathcal{R}F - s f_1 - (c_1 - f_1) b \phi_2 = (c_1 - f_1)\Phi + \varepsilon G(\Phi, s, b; \varepsilon), \\ &\langle \phi_1, F \rangle = 0, \\ &\langle \phi_2, F \rangle = 0, \end{aligned}$$

or

$$M \begin{pmatrix} F \\ s \\ b \end{pmatrix} = \begin{pmatrix} (c_1 - f_1)\Phi + \varepsilon G(F, s, b; \varepsilon) \\ 0 \\ 0 \end{pmatrix}, \quad M := \begin{pmatrix} \mathcal{R} & -f_1 & -(c_1 - f_1)\phi_2 \\ \langle \phi_1, \cdot \rangle & 0 & 0 \\ \langle \phi_2, \cdot \rangle & 0 & 0 \end{pmatrix}.$$

We now express this as $x = T(x)$ where

$$x := \begin{pmatrix} F \\ s \\ b \end{pmatrix}, \quad T(x) := M^{-1} \begin{pmatrix} (c_1 - f_1)\Phi + \varepsilon G(F, s, b; \varepsilon) \\ 0 \\ 0 \end{pmatrix},$$

and seek a fixed point of the iteration

$$x_n = T(x_{n-1}).$$

From Theorem 3 we know that M^{-1} is well–defined and that the action of $x^* = T(x)$ can be given by

$$s^* = -\frac{1}{2}\left(Q_1(F, s, b) + \frac{Q_2(F, s, b)}{\beta} \right), \tag{17a}$$

$$b^* = -\frac{1}{2}\left(-2Q_1(F, s, b) + \frac{2Q_2(F, s, b)}{\beta} \right), \tag{17b}$$

$$F^* = \mathcal{R}^{-1}\left[s^* f_1 + (c_1 - f_1) b^* \phi_2 + (c_1 - f_1)\Phi + \varepsilon G(F, s, b; \varepsilon) \right], \tag{17c}$$

where

$$Q_m(F, s, B) := \langle (c_1 - f_1)\Phi + \varepsilon G(F, s, b; \varepsilon), \psi_m \rangle, \quad m = 1, 2, , \tag{17d}$$

c.f. (15) and (16). A fixed point of the iteration $x_n = T(x_{n-1})$ would solve $x = T(x)$ and represent a solution of (2). To achieve this we appeal to Banach's Fixed Point Theorem [30] which requires a Banach space,

$$X = H^{\sigma+\tau} \times \mathbf{R} \times \mathbf{R},$$

and a distance,

$$d(x, y)^2 = \|F_x - F_y\|^2_{H^{\sigma+\tau}} + |s_x - s_y|^2 + |b_x - b_y|^2.$$

We are now in a position to prove the central estimate which establishes that T is a contraction for ε small enough.

Lemma 4 *Provided that $\sigma > 1/2$ and $x, y \in B_R(0) \subset X$ we have that*

$$d(T(x), T(y))^2 \le \varepsilon^2 C(R)^2 d(x, y)^2,$$

where $C(R)$ is a quadratic function of R.

Proof Exactly analogous to that of Lemma 2.

Just as we did for the Stokes waves, using Lemma 4 we can show that, for sufficiently small ε, if the sequence $x_n = T(x_{n-1})$ starts in $B_R(0)$ it always stays there, c.f. Lemma 3.

Lemma 5 *Provided that $\sigma > 1/2$, if $x_0 \in B_R(0)$ and $x_n = T(x_{n-1})$ then, for any $R > 0$, there exists $\varepsilon > 0$ such that $x_n \in B_R(0)$ for all $n \ge 0$.*

This results in our final theorem which has the same proof as that of the Stokes result, Theorem 2.

Theorem 4 *Provided that $\sigma > 1/2$, for ε sufficiently small the mapping T is a contraction on $B_R(0)$ implying that, for ε sufficiently small, there exists a unique Wilton Ripple of the form* (9).

Model	Operator	c_0	Spectral Gap	Asymptotic	ε^+	ε^-
(A) Whitham (Deep Water)	$\hat{\mathcal{L}} = \sqrt{\lvert k\rvert^{-1} + \frac{1}{2}\lvert k\rvert}$	$\sqrt{3/2}$	$\sqrt{\frac{11}{6}} - \sqrt{\frac{3}{2}} \approx 0.129$	$\mathcal{O}(k^{1/2})$	0.14	0.22
(B) Akers–Milewski	$\hat{\mathcal{L}} = \lvert k\rvert^{-1} + \frac{1}{2}\lvert k\rvert$	3/2	1/3	$\mathcal{O}(k)$	0.38	0.61
(C) Benjamin	$\hat{\mathcal{L}} = \lvert k\rvert - \frac{1}{3}\lvert k\rvert^2$	2/3	2/3	$\mathcal{O}(k^2)$	1.48	0.96
(D) Hyper–Akers–Milewski	$\hat{\mathcal{L}} = (\lvert k\rvert^{-1} + \frac{1}{2}\lvert k\rvert)^2$	9/4	≈ 1.1	$\mathcal{O}(k^2)$	1.44	2.30
(E) Kawahara	$\hat{\mathcal{L}} = k^2 - \frac{1}{5}k^4$	4/5	8	$\mathcal{O}(k^4)$	3.33	5.10
(F) Hyper–Whitham	$\hat{\mathcal{L}} = (\lvert k\rvert^{-1} + \frac{1}{6}\lvert k\rvert^2)^2$	49/36	2	$\mathcal{O}(k^4)$	22.37	15.36

Table 1 Listing of key parameters of the six models simulated in this paper: The Fourier symbol of the linear operator, the phase speed at which there are Wilton Ripples (between wavenumbers $k = 1$ and $k = 2$), the spectral gap, the high frequency asymptotics, and an estimate for the maximal ε for which each branch converged for each model.

5 Numerical Results

Inspired by the above results, we have implemented a numerical algorithm based on the FPI (14) which, we note, converges only for amplitudes up to a *finite* value, beyond which the iteration ceases to be a contraction. Due to the flexibility of not only our formulation but also our numerical implementation, we readily computed traveling Wilton Ripples for *four* model PDEs which appear in the literature, the Kawaraha [40], Benjamin [15], Whitham [47], and Akers–Milewski [2] equations. In addition, we simulated two model PDEs which also fit into our framework that appear for the first time here, the Hyper–Akers–Milewski and Hyper–Whitham equations. We refer the reader to Table 1 for a detailed prescription of each of these six PDEs, in particular the "Spectral Gap" that we define as

$$\min_{k>2} \left| \mathcal{L}(k) - \mathcal{L}(1) \right|,$$

which is significant as the radii R from Theorems 2 and 4 tend to zero as this gap vanishes.

Regarding our novel numerical FPI algorithm, we implemented a Fourier collocation approach [32,18,16,54,55] where the unknown functions, $F_n(x)$, in (14) were approximated by

$$F_n(x) \approx F_n^{N_x}(x) = \sum_{k=-N_x/2}^{N_x/2-1} a_{n,k} e^{ikx}, \quad a_{n,k} \approx \frac{1}{2\pi} \int_0^{2\pi} F_n(x) e^{-ikx}\, dx,$$

and (14) was enforced at the equally–spaced gridpoints $\{x_j = 2\pi j/N_x\}$ for $0 \le j \le N_x - 1$. We began by choosing $\{c_0, \phi_1, \phi_2, \Phi, c_1, \beta\}$ as outlined in § 4.1 and § 4.2, and initiated our iteration with $\{F_0, s_0, b_0\} = \{0, 0, 0\}$. For $n \ge 1$ we conducted the iteration (14) which involved two steps: Utilizing (15) to determine the next $\{s_n, b_n\}$ iterates, and then solving (16) to find $F_n^{N_x}$ via the Discrete Fourier Transform (DFT)—accelerated by the Fast Fourier Transform (FFT) algorithm—which diagonalizes the operator $\mathcal{R}$. In these formulas derivatives and Fourier multipliers were evaluated in Fourier space with the

DFT, and products were implemented in physical space; we did not notice appreciable errors due to aliasing. This iteration was continued until the Cauchy error,

$$e_N := \|f_N - f_{N-1}\|_{L^2} + |s_N - s_{N-1}| + |b_N - b_{N-1}|\,, \tag{18}$$

reached machine precision ($\approx 2 \times 10^{-16}$).

We compared the results of this novel iteration scheme with a classic Quasi–Newton iteration of (2) given the quadratic Wilton Ripple as a starting guess. More specifically, we once again used a Fourier collocation method [32,18,16, 54,55] where the unknown function, $F(x)$, in (2) was approximated by

$$F(x) \approx F^{N_x}(x) = \sum_{k=-N_x/2}^{N_x/2-1} a_k e^{ikx}, \quad a_k \approx \frac{1}{2\pi} \int_0^{2\pi} F(x) e^{-ikx}\, dx,$$

and a numerical continuation approach [12] applied to (2) with an initial guess provided by (13) with $\{F, s, b\}$ set to zero. We note that both approaches produced the same branches of traveling Wilton Ripples, provided that they converged. While the Quasi–Newton solver converged for larger solutions, it was far more expensive due to the *formation* and *factorization* of the *full* Jacobian matrix. By contrast, our FPI was much more rapid, though its utility for larger solutions was more limited.

In Figure 1 we display results of our novel iteration scheme for the numerical simulation of (2) for the case of the Akers–Milewski equation (B in Table 1). On the left we display two traveling wave profiles, the solid curve corresponding to $\beta = 1/\sqrt{2}$, c.f. (12), and the dashed curve comes from $\beta = -1/\sqrt{2}$. The speed–amplitude curve for each of these two branches is plotted on the right of this figure with both our FPI scheme and the Quasi–Newton approach [20]. The FPI converges for a finite range of amplitudes which we denote as $\varepsilon^{\pm}$ corresponding to $\beta = \pm 1/\sqrt{2}$. We estimated these amplitudes for the six PDEs listed in Table 1 and report the results there.

We close with results on the estimated errors in our computations and the time required for our new FPI versus the classical Quasi–Newton method. For the error, as there is no exact solution, we settled on the Cauchy measure, (18). On the left panel in Figure 2 we plot this versus iteration number N, while on the right of Figure 2 we display run times for the FPI (triangles) and the Quasi–Newton method (circles) as a function of the number of spatial points, N_x. Over this range of points the cost of the Quasi–Newton method is dominated by the formation of the Jacobian matrix, $\mathcal{O}(N_x^2 \log N_x)$, while the FPI has computational complexity $\mathcal{O}(N_x \log N_x)$ (these asymptotic curves are depicted with solid lines in the figure).

6 Conclusion

In this paper we have undertaken the study of weakly nonlinear dispersive wave equations which admit Wilton Ripple solutions. Not only do we demonstrate their existence rigorously using a Banach Fixed Point Iteration, but we

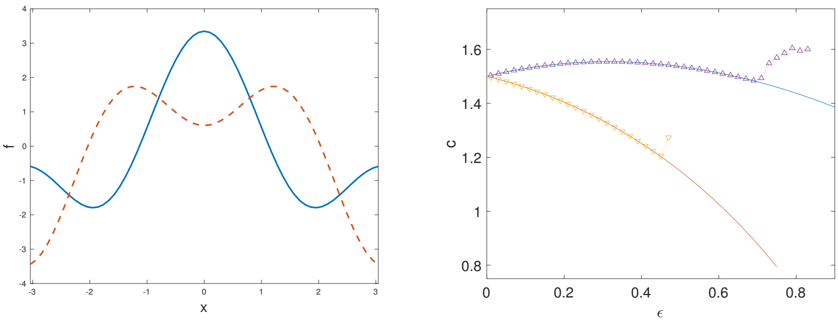

Fig. 1 **Left:** Example profiles from the FPI, (14), with $\varepsilon = 0.01$ (the profile on branch with $\beta = 1/\sqrt{2}$, (12), is the solid curve, the profile on the branch with $\beta = -1/\sqrt{2}$ is the dashed curve). **Right:** The speed–amplitude curves for two branches of ripples in the Akers–Milewski equation are depicted. The output of the FPI (triangles) is compared to that of the Quasi–Newton iteration (solid lines).

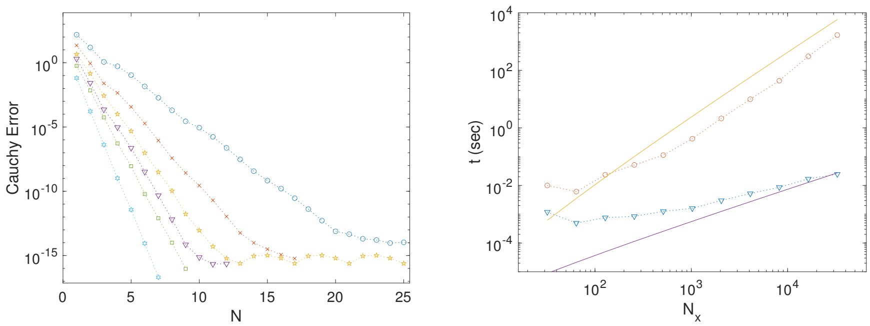

Fig. 2 **Left:** The Cauchy error, (18), measured for a Wilton Ripple in each of the PDEs listed in Table 1 from the FPI (14) at $\varepsilon = 0.01$. The markers correspond to the PDEs as follows: (A) circles, (B) x's, (C) stars, (D) triangles, (E) squares, (F) hexagons. **Right:** The run times for our novel FPI (triangles) and the Quasi–Newton method (circles) as a function of the number of spatial points. The amplitude parameter $\varepsilon = 0.01$ was simulated for the Akers–Milewski equation. The asymptotic curves $N_x^2 \log N_x$ and $N_x \log N_x$ are plotted with solid lines.

also utilize a numerical simulation of this iteration scheme to produce highly accurate solutions in a fraction of the cost of a classical Quasi–Newton approach.

The Wilton Ripple problem has very similar structure to that of the spectral stability problem near an eigenvalue collision. This similarity suggests that very similar methods to those of this paper can be used to prove their existence and compute instabilities. The tools of this paper do not require analyticity (in contrast to our previous work [7]) and thus have the potential to compute instabilities which are merely continuous, e.g., of Benjamin–Feir type.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

Data will be made available on reasonable request.

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