Water Waves
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ORIGINAL ARTICLE

Wilton Ripples in Weakly Nonlinear Dispersive Models of Water Waves: Existence and Analyticity of Solution Branches

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Abstract

Traveling waves on the surface of the ocean play an important role in many oceanographic processes which necessitates a detailed quantitative understanding of their properties. The water wave equations, which govern the free-surface evolution of an ideal fluid, are the most successful model for this phenomena, but are exceedingly difficult to analyze due to their strongly nonlinear character and the fact that they are posed on a domain with moving boundary. For this reason, weakly nonlinear dispersive models are an essential tool for practitioners, and in this contribution, we study traveling wave solutions of a broad class of such models. The simplest family of traveling wave solutions are the Stokes waves which can be characterized as simple bifurcation (one-dimensional null space of the linearized operator) from the trivial (flat-water) branch of solutions. We focus our analysis on the much less studied non-simple case of Wilton ripples which have linear behavior characterized by two co-propagating harmonics (a two-dimensional null space of the linearized operator). More specifically, we show that such branches of solutions exist for a class of nonlinear dispersive model equations, and that they are analytic with respect to a natural wave height/slope parameter.

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1 Introduction

The motion of waves on the surface of a large body of water (e.g., a lake or ocean) plays an important role in applications from many branches of science and engineering. For instance, from the design of oil rigs [31], to following the motion of underwater sandbars [43,44], to the propagation of tsunamis across the ocean [10], examples abound. The most successful model for the motion of these surface water waves are the water wave equations [1,29,47] which are a challenging system of partial differential equations of both theoretical and practical interest. Despite their compelling modeling capabilities, the water wave equations are extremely difficult to solve due not only to the property that the upper boundary of the problem domain is free (moving), but also that the governing equations of this motion are strongly nonlinear. For this reason, model equations valid in various scaling regimes have played an important role in the study of water waves for well over a century. The Korteweg–de Vries (KdV) and Nonlinear Schrödinger equation (NLS) are well-known examples [57], but many others exist and in this contribution we will utilize several of these.

Among solutions to the water wave equations, the traveling waves are distinguished by their practical importance in efficiently transporting energy, momentum, and pollutants across great distances [26,50,54]. Any reasonable model of these equations also accommodates traveling wave solutions and our purpose in this work is to study families of such solutions in the presence of resonance. It is well known that spatially periodic families of traveling water waves exist whose interface shape features a single wavelength in the linear limit [48]. These "Stokes waves" have been found in many model equations as well, and constitute a readily analyzed instance of simple bifurcation [15] (one-dimensional null space of the linearized operator) from the trivial (flat-interface) solution.

Stokes waves have a significant history of both asymptotic [14,20,32,48] and numerical study [5,12,37,39,45,46]. The rigorous treatment of Stokes waves is also well developed: they are known to exist [30,49] and, in particular connection to the current contribution, be parametrically analytic in amplitude/slope [11,38]. The global bifurcation problem has been described and extreme waves on branches have been characterized [6,51].

Given the existence of such solution branches one can ponder the existence of bifurcation in the context of a two (or higher) dimensional null space. Particular examples of this are the "Wilton ripples" [58] which have been found in the water wave equations with non-zero surface tension at critical values, and exist for *some* of their model equations. At these critical values of the surface tension, the linear operator has a two-dimensional kernel and the linear problem supports two co-propagating harmonics of the form $\cos(nx)$ and $\cos(mx)$. Continuous branches of small amplitude waves exist about special ratios of these harmonics, and these Wilton ripples have been studied for over a century now, including asymptotic [21,24,34,58], numerical [3,7,9,55], and experimental work [22,33,40].

The rigorous treatment of the Wilton ripple problem is less complete than that of the Stokes wave. Most studies treat the resonant (one of n and m is an integer multiple of the other) and non-resonant (neither n nor m is an integer multiple of the other) two-dimensional null space separately, due to the change in form of the wave's asymptotics. The bifurcation structure has been described in the absence of harmonic resonance for both potential flow [2] and the hydroelastic water wave problem [23], while the corresponding description for the Whitham equation has recently appeared in [16]. Of greatest relevance to the current study is the work of Reeder and Shinbrot. These authors informally discussed the phenomena of resonant Wilton ripples for the full water wave problem in [41], while rigorously demonstrating their existence and analyticity in [42] using an iteration scheme in a Banach space. This is the only work of which we are aware on the resonant configuration we address here. The stability of these ripples has been investigated in the hydroelastic [53] and water wave problems [52].

In contrast to this, in the current contribution we produce a proof of existence and analyticity by the "Method of Majorants" [18,39] which justifies the perturbative approach to the problem in the spirit of the author and Reitich in [38] for the full water wave equations for Stokes waves. While we cannot yet accomplish this analysis for the full water wave system, we do present a rigorous analysis for a class of weakly nonlinear model equations which include the Kawahara [25], Benjamin [8], Whitham [35], and Akers–Milewski [4] equations. With this in place, we postulate that a similar result for the full set of equations is within reach, and this is the object of current investigation by the authors.

The rest of the paper is organized as follows. We present the governing equations in Sect. 2 and define relevant function spaces in Sect. 3. We make a careful study of the linearized problem in Sect. 4 which is central to our bifurcation theory. We study the existence and analyticity of branches of Stokes waves in Sect. 5 and their Wilton ripple counterparts in Sect. 6. We close with concluding remarks in Sect. 7.

2 Governing Equations

As we mentioned above, in this work, we focus on laterally (x) periodic solutions of weakly nonlinear dispersive wave equations, e.g.,

$$\partial_t u + \mathcal{L}\partial_x u - N(u) = 0, \quad u(x + 2\pi, t) = u(x, t),$$

meant to be model equations for the full water wave equations in the limit of long wavelengths and weak nonlinearity. While many famous equations fit into this generic framework, we focus upon those which support co-propagating traveling wave solutions in the linear limit. Examples of this are the Kawahara [25],

$$\partial_t u + \left[D^2 - \frac{1}{5} D^4 \right] \partial_x u - \frac{1}{2} u \partial_x u = 0,$$

Benjamin [8],

$$\partial_t u + \left[|D| - \frac{1}{3} |D|^2 \right] \partial_x u - \frac{1}{2} u \partial_x u = 0,$$

(Deep Water) Whitham [35],

$$\partial_t u + \left[\sqrt{|D|^{-1} + \frac{1}{2} |D|} \right] \partial_x u - \frac{1}{2} u \partial_x u = 0,$$

and Akers–Milewski [4]

$$\partial_t u + \left[|D|^{-1} + \frac{1}{2} |D| \right] \partial_x u - \frac{1}{2} u \partial_x u = 0,$$

equations. In these, the Fourier multiplier $m(D)$ is understood from the definition

$$m(D) \left[u(x, t) \right] := \sum_{p=-\infty}^{\infty} m(p) \hat{u}_p(t) e^{ipx},$$

where

$$u(x, t) = \sum_{p=-\infty}^{\infty} \hat{u}_p(t) e^{ipx}, \quad \hat{u}_p(t) = \frac{1}{2\pi} \int_0^{2\pi} u(y, t) e^{-ipy} \, \mathrm{d}y.$$

So, e.g., $\partial_x = iD$ as

$$\partial_x u(x, t) = \sum_{p=-\infty}^{\infty} (ip) \hat{u}_p(t) e^{ipx} = iDu(x, t).$$

All of these equations share a common nonlinearity,

$$N(u) = \frac{1}{2} u \partial_x u = \partial_x \left[u^2 \right],$$

whose form is convenient, but not essential, for our developments. For simplicity, we focus on weakly nonlinear dispersive equations of the form

$$\partial_t u + \mathcal{L} \partial_x u - \partial_x \left[u^2 \right] = 0, \tag{1a}$$

where

$$\mathcal{L}(D) = \begin{cases} D^2 - (1/5) D^4, & \text{Kawahara}, \\ |D| - (1/3) |D|^2, & \text{Benjamin}, \\ \sqrt{|D|^{-1} + (1/2) |D|}, & \text{Whitham}, \\ |D|^{-1} + (1/2) |D|, & \text{Akers–Milewski}, \end{cases} \tag{1b}$$

and, from above,

$$|D|\,u(x,t) = \sum_{p=-\infty}^{\infty} |p|\,\hat{u}_p(t)e^{ipx}.$$

At this point, we turn to the object of our study, traveling wave solutions of (1)

$$u(x,t) = f(x-ct),$$

which satisfy

$$-c\partial_x f + \mathcal{L}\partial_x f - \partial_x\left[f^2\right] = 0.$$

Upon integrating once with respect to x it is easy to see that traveling wave solutions satisfy

$$[-c+\mathcal{L}]\,f = f^2. \tag{2}$$

We have dropped the arbitrary constant of integration which constitutes a zero-mean assumption.

3 Function Spaces

Before beginning our rigorous analysis, we recall the relevant function spaces we require. For any real $\sigma \geq 0$, we have the classical L^2-based Sobolev norm [27]

$$\|f\|_{H^\sigma}^2 := \sum_{p=-\infty}^{\infty} \langle p\rangle^{2\sigma}\left|\hat{f}_p\right|^2, \quad \langle p\rangle^2 := 1+|p|^2,$$

which gives rise to the periodic Sobolev space [27]

$$H^\sigma([0,2\pi]) = \left\{f(x)\in L^2([0,2\pi]) \mid \|f\|_{H^\sigma} < \infty\right\}.$$

The dual of H^σ is $H^{-\sigma}$ which can be equivalently defined with the negative index norm above. With sufficient regularity, these Sobolev spaces satisfy a well-known algebra property [17,28].

Lemma 1 *If $\sigma > 1/2$ and $f, g \in H^\sigma$ then $fg \in H^\sigma$ and there is a constant M such that*

$$\|fg\|_{H^\sigma} \leq M\,\|f\|_{H^\sigma}\,\|g\|_{H^\sigma}.$$

To close, we note the following mapping properties of the linearized operators $[-c+\mathcal{L}]$ which are readily verified:

$$[-c+\mathcal{L}] : H^{\sigma+\tau} \to H^\sigma, \tag{3a}$$

where

$$\tau = \begin{cases} 4, & \text{Kawahara}, \\ 2, & \text{Benjamin}, \\ 1/2, & \text{Whitham}, \\ 1, & \text{Akers–Milewski}. \end{cases} \tag{3b}$$

These operators have inverses for functions in their ranges

$$Y^\sigma = \left\{ f \in H^\sigma \mid f \in \text{ran}([-c + \mathcal{L}]) \right\},$$

and they "give back" τ-many derivatives

$$[-c + \mathcal{L}]^{-1} : Y^\sigma \to H^{\sigma+\tau}. \tag{4}$$

4 The Linearized Problem

Bifurcation theory [15] tells us that non-trivial *branches* of solutions bifurcate from the trivial (flat-interface) branch at values of the parameter c where the linearized operator $[-c + \mathcal{L}]$ has a non-trivial null space. We are particularly interested in the case of non-simple bifurcation where the null space has dimension greater than one. More specifically, for the four equations listed above, it is the case that there exists a c_0 such that

$$\mathcal{N}([-c_0 + \mathcal{L}]) = \text{span}(\phi_1(x), \phi_2(x)), \quad \phi_m(x) = \cos(mx),$$

and

$$\mathcal{N}([-c_0 + \mathcal{L}]^*) = \text{span}(\psi_1(x), \psi_2(x)), \quad \psi_m(x) = \cos(mx),$$

so that each supports *two* ($m = 1, 2$) co-propagating harmonics in the linearized problem. It is not difficult to see that the appropriate choices of c_0 are

$$c_0 = \begin{cases} 4/5, & \text{Kawahara}, \\ 2/3, & \text{Benjamin}, \\ \sqrt{3/2}, & \text{Whitham}, \\ 3/2, & \text{Akers–Milewski}. \end{cases}$$

We now rewrite (2) a little

$$[-c_0 + \mathcal{L}]\, f = (c - c_0) f + f^2, \tag{5}$$

and seek small solutions $f = \mathcal{O}(\varepsilon)$ and $(c - c_0) = \mathcal{O}(\varepsilon)$ as functions of the height/slope parameter ε. At order $\mathcal{O}(\varepsilon)$ we find that (5) demands

$$[-c_0 + \mathcal{L}]\, f_1 = 0,$$

where $f = \varepsilon f_1 + \mathcal{O}(\varepsilon^2)$. As we mentioned above, in order to make any progress it must be the case that the "resolvent"

$$\mathcal{R} := -c_0 + \mathcal{L} = \mathcal{O}(1),$$

has a non-trivial null space and we will choose f_1 to lie here. Finally, we note that, for the inner product

$$\langle u, v \rangle = \int_0^{2\pi} u(x)v(x)\, \mathrm{d}x,$$

we have

$$\langle \phi_m, \phi_\ell \rangle = \pi \delta_{m,\ell}, \quad \langle \psi_m, \psi_\ell \rangle = \pi \delta_{m,\ell}, \quad \langle \phi_m, \psi_\ell \rangle = \pi \delta_{m,\ell},$$

where $\delta_{m,\ell}$ is the Kronecker delta function.

5 Stokes Waves

To introduce notation and our methodology we begin our developments with the well-understood Stokes waves. This also serves to describe a straightforward machinery against which to contrast the more subtle developments required for the Wilton ripples in Sect. 6. The scenario of Stokes waves is one of simple bifurcation [15] with a one-dimensional null space

$$\mathcal{N}(\mathcal{R}) = \mathrm{span}(\phi), \quad \mathcal{N}(\mathcal{R}^*) = \mathrm{span}(\psi), \quad \phi(x) = \psi(x) = \cos(x).$$

With these, we specify

$$f_1 = \alpha\phi = \phi,$$

where we have set $\alpha = 1$ without loss of generality.

We identify our unknowns: the *correction* of the speed and interface profile,

$$s := c - c_0 = \mathcal{O}(\varepsilon), \tag{6a}$$

$$F := f - \varepsilon f_1 = \mathcal{O}(\varepsilon^2), \tag{6b}$$

and write the governing equations, (5), as

$$\mathcal{R}[f] = sf + f^2 = (s + f)f. \tag{7}$$

Using the fact that $\phi \in \mathcal{N}(\mathcal{R})$, this simplifies to

$$\mathcal{R}[F] = (s + \varepsilon\phi + F)(\varepsilon\phi + F) = \left\{\varepsilon^2\phi^2 + \varepsilon\phi s\right\} + \{2\varepsilon\phi F + sF\} + \left\{F^2\right\}, \tag{8}$$

which groups terms of orders $\mathcal{O}(\varepsilon^2)$, $\mathcal{O}(\varepsilon^3)$, and $\mathcal{O}(\varepsilon^4)$, respectively. We make the expansions

$$s(\varepsilon) = \sum_{n=1}^{\infty} s_n \varepsilon^n, \tag{9a}$$

$$F(x;\varepsilon) = \sum_{n=2}^{\infty} F_n(x) \varepsilon^n, \tag{9b}$$

and insert these into (8) to find

$$\mathcal{R}\left[F_n\right] = \delta_{n,2}\phi^2 + \phi s_{n-1} + \mathcal{Q}_n, \tag{10}$$

where

$$\mathcal{Q}_n = (2\phi + s_1)F_{n-1} + \sum_{\ell=2}^{n-2} s_{n-\ell} F_\ell + \sum_{\ell=2}^{n-2} F_{n-\ell} F_\ell. \tag{11}$$

5.1 Second Order

To initiate our perturbation approach, we begin with the special case at order $\mathcal{O}(\varepsilon^2)$ where the governing equations (10) become

$$\mathcal{R}\left[F_2\right] = s_1\phi + \phi^2.$$

The unknowns are $\{F_2, s_1\}$ and to resolve this equation we begin by demanding solvability. We can ensure this by projecting onto ψ,

$$0 = \langle \mathcal{R}\left[F_2\right], \psi \rangle = \left\langle s_1\phi + \phi^2, \psi \right\rangle = s_1 \langle \phi, \psi \rangle + \left\langle \phi^2, \psi \right\rangle,$$

so that

$$s_1 = -\left\langle \phi^2, \psi \right\rangle / \langle \phi, \psi \rangle = -\left\langle \phi^2, \psi \right\rangle = 0,$$

where we have used the readily computed facts that $\left\langle \phi^2, \psi \right\rangle = 0$ and $\langle \phi, \psi \rangle = 1$. One can now discover F_2 from

$$\mathcal{R}\left[F_2\right] = \phi^2 - \left\langle \phi^2, \psi \right\rangle \phi,$$

coupled to the demand that $\langle F_2, \phi \rangle = 0$.

5.2 Analyticity

We are now in a position to move on to the existence of traveling wave solutions and the analyticity of the branch of these solutions with respect to the wave height/slope parameter, ε. The key to this effort is the following generic elliptic estimate.

Lemma 2 *For any real $\sigma \geq 0$, given $Q \in H^{\sigma}$ there exists a unique solution pair $\{F, s\}$ of*

$$\mathcal{R}F - s\phi = Q, \tag{12}$$

satisfying

$$\max \{\|F\|_{H^{\sigma+\tau}}, |s|\} \leq C_e \|Q\|_{H^{\sigma}},$$

for some universal constant $C_e > 0$.

Proof We begin by ensuring solvability which we guarantee by choosing s appropriately. Taking the inner product of (12) with ψ yields

$$-s \langle \phi, \psi \rangle = \langle \mathcal{R}F - s\phi, \psi \rangle = \langle Q, \psi \rangle,$$

so that

$$s = -\langle Q, \psi \rangle / \langle \phi, \psi \rangle = -\frac{1}{\pi} \langle Q, \psi \rangle.$$

Using the Cauchy–Schwartz inequality we find

$$|s| \leq \frac{1}{\pi} |\langle Q, \psi \rangle| \leq \frac{1}{\sqrt{\pi}} \|Q\|_{L^2} \leq \frac{1}{\sqrt{\pi}} \|Q\|_{H^{\sigma}}.$$

With this definition of s we readily discover that

$$\mathcal{R}F = Q + s\phi \in \operatorname{ran}(\mathcal{R}),$$

so that the estimate on F is clearly true from the mapping properties of $\mathcal{R}$, (3) and (4). □

With this, we can establish a recursive estimate which is fundamental to our analyticity result.

Lemma 3 *For any real $\sigma > 1/2$, if*

$$\|F_n\|_{H^{\sigma+\tau}} \leq C_F \frac{D^{n-2}}{(n+1)^2}, \qquad 2 \leq n \leq N-1,$$

$$|s_{n-1}| \leq C_s \frac{D^{n-2}}{(n+1)^2}, \qquad 2 \leq n \leq N-1,$$

for $N \geq 3$, then we have

$$\|\mathcal{Q}_N\|_{H^\sigma} \leq C_F \tilde{C} \frac{D^{N-3}}{(N+1)^2}, \quad N \geq 3,$$

c.f. (11), for some $\tilde{C} > 0$.

Proof We begin with the simple estimate (which requires $\sigma > 1/2$)

$$\begin{aligned}
\|\mathcal{Q}_N\|_{H^\sigma} &\leq \left(2M \|\phi\|_{H^\sigma} + |s_1|\right) \|F_{N-1}\|_{H^\sigma} \\
&\quad + \sum_{\ell=2}^{N-2} |s_{N-\ell}| \, \|F_\ell\|_{H^\sigma} + \sum_{\ell=2}^{N-2} M \, \|F_{N-\ell}\|_{H^\sigma} \, \|F_\ell\|_{H^\sigma} \\
&\leq \left(2M \|\phi\|_{H^\sigma} + |s_1|\right) C_F \frac{D^{N-3}}{(N-1+1)^2} \\
&\quad + \sum_{\ell=2}^{N-1} C_s \frac{D^{N-\ell-1}}{(N-\ell+1)^2} C_F \frac{D^{\ell-2}}{(\ell+1)^2} \\
&\quad + \sum_{\ell=2}^{N-2} M C_F \frac{D^{N-\ell-2}}{(N-\ell+1)^2} C_F \frac{D^{\ell-2}}{(\ell+1)^2} \\
&\leq C_F \left(2M \|\phi\|_{H^\sigma} + |s_1|\right) \frac{D^{N-3}}{(N-1+1)^2} \\
&\quad + C_F C_s S \frac{D^{N-3}}{(N+1)^2} + C_F C_F M S \frac{D^{N-4}}{(N+1)^2},
\end{aligned}$$

and we are done provided that

$$\tilde{C} = 2M \|\phi\|_{H^\sigma} + |s_1| + C_s S + C_F M S.$$

□

Now we can state and prove our existence and analyticity result for Stokes waves.

Theorem 1 *For any real $\sigma > 1/2$*

$$\|F_n\|_{H^{\sigma+\tau}} \leq C_F \frac{D^{n-2}}{(n+1)^2}, \quad |s_{n-1}| \leq C_s \frac{D^{n-2}}{(n+1)^2}, \quad n \geq 3, \tag{13}$$

for some $C_F, C_s > 0$.

Proof We work by induction in n. The base case $n = 2$ is straightforward, and we now suppose that (13) is true for all $n < N$, for $N \geq 3$. Using Lemma 2, we estimate

$$\max\{\|F_N\|_{H^{\sigma+\tau}}, s_{N-1}\} \leq C_e \|\mathcal{Q}_N\|_{H^\sigma},$$

and from Lemma 3 we find

$$\max\{\|F_N\|_{H^{\sigma+\tau}}, s_{N-1}\} \leq C_e C_F \tilde{C} \frac{D^{N-3}}{(N+1)^2}.$$

The estimate on F_N requires

$$C_e C_F \tilde{C} \frac{D^{N-3}}{(N+1)^2} \leq C_F \frac{D^{N-2}}{(N+1)^2},$$

which we can guarantee if

$$D > C_e \tilde{C}.$$

Meanwhile, the bound on $|s_{N-1}|$ demands that

$$C_e C_F \tilde{C} \frac{D^{N-3}}{(N+1)^2} \leq C_s \frac{D^{N-2}}{(N+1)^2},$$

which is possible if

$$D \geq C_e (C_F / C_s) \tilde{C}.$$

□

For emphasis, we state the implication of this theorem as a corollary.

Corollary 1 *The functions $s(\varepsilon)$ and $F(x;\varepsilon)$ of (6) are real analytic functions of ε in a sufficiently small neighborhood of the origin in the complex plane. More precisely, for any $0 \leq \rho < 1$, the Taylor series expansions*

$$s(\varepsilon) = \sum_{n=1}^{\infty} s_n \varepsilon^n, \quad F(x;\varepsilon) = \sum_{n=2}^{\infty} F_n(x) \varepsilon^n,$$

c.f. (9), converge for all ε such that $D\varepsilon < \rho$, i.e., $\varepsilon < \rho/D$.

6 Wilton Ripples

We are now in a position to prove our primary result, the existence and analyticity of branches of Wilton ripple solutions. In this case ,we recall that

$$\mathcal{N}(\mathcal{R}) = \mathrm{span}(\phi_1, \phi_2), \quad \mathcal{N}(\mathcal{R}^*) = \mathrm{span}(\psi_1, \psi_2),$$
$$\phi_m(x) = \psi_m(x) = \cos(mx).$$

Here we build upon the success of the Stokes wave case and choose

$$s := c - c_0 = \mathcal{O}(\varepsilon), \quad f_1 = \phi_1 + \beta\phi_2; \quad \beta \in \mathbf{R}; \tag{14a}$$
$$f(x;\varepsilon) = \varepsilon f_1(x) + B(x;\varepsilon) + F(x;\varepsilon); \quad B, F = \mathcal{O}(\varepsilon^2), \tag{14b}$$

where

$$B(x;\varepsilon) = b(\varepsilon)\phi_2(x), \quad \langle F, \phi_1 \rangle = \langle F, \phi_2 \rangle = 0.$$

We point out that this approach requires a slightly different mindset as the correction of the linear solution now involves both a member of the null space, $B(x;\varepsilon)$, as well as a component in its orthogonal complement, $F(x;\varepsilon)$.

We recall our governing equations (7),

$$\mathcal{R}[f] = sf + f^2 = (s+f)f,$$

and insert our form above into this giving

$$\begin{aligned}\mathcal{R}[F] &= (s + \varepsilon f_1 + B + F)(\varepsilon f_1 + B + F) \\ &= \{\varepsilon f_1(s + \varepsilon f_1)\} + \{\varepsilon f_1(2B + 2F) + s(B+F)\} + \left\{(B+F)^2\right\},\end{aligned}$$

where we have used the fact that f_1 and B are both in the null space of $\mathcal{R}$, and grouped terms which are $\mathcal{O}(\varepsilon^2)$, $\mathcal{O}(\varepsilon^3)$ and $\mathcal{O}(\varepsilon^4)$, respectively. Upon inserting our expansions, (9) supplemented with

$$B(x;\varepsilon) = \sum_{n=2}^{\infty} B_n(x)\varepsilon^n = \sum_{n=2}^{\infty} b_n\phi_2(x)\varepsilon^n,$$

we have, for $n = 2$,

$$\mathcal{R}[F_2] = f_1^2 + s_1 f_1 \tag{15}$$

and, for $n \geq 3$,

$$\mathcal{R}[F_n] = f_1 s_{n-1} + (2f_1 + s_1)\, B_{n-1} + \mathcal{Q}_n, \tag{16}$$

c.f., (10),

$$\mathcal{Q}_n = (2f_1 + s_1)\, F_{n-1} + \sum_{\ell=2}^{n-2} (s_{n-\ell} + B_{n-\ell} + F_{n-\ell})(B_\ell + F_\ell), \tag{17}$$

c.f., (11).

6.1 Second Order

As before, to begin our perturbative method, we examine the second-order term where we have

$$\mathcal{R}[F_2] = s_1 f_1 + f_1^2,$$

c.f. (15). Here the unknowns are $\{F_2, s_1, \beta\}$ and we determine the latter two by demanding solvability. This we accomplish by projecting onto ψ_1 and ψ_2 resulting in

$$0 = \langle \mathcal{R}[F_2], \psi_1 \rangle = \left\langle s_1(\phi_1 + \beta\phi_2) + (\phi_1 + \beta\phi_2)^2, \psi_1 \right\rangle$$
$$0 = \langle \mathcal{R}[F_2], \psi_2 \rangle = \left\langle s_1(\phi_1 + \beta\phi_2) + (\phi_1 + \beta\phi_2)^2, \psi_2 \right\rangle,$$

or

$$C_{11}s_1 + C_{21}s_1\beta + C_{111} + 2C_{121}\beta + C_{221}\beta^2 = 0,$$
$$C_{21}s_1 + C_{22}s_1\beta + C_{112} + 2C_{122}\beta + C_{222}\beta^2 = 0,$$

where

$$C_{jm} = \left\langle \phi_j, \psi_m \right\rangle, \quad C_{j\ell m} = \left\langle \phi_j\phi_\ell, \psi_m \right\rangle.$$

The solvability of these nonlinear equations appears highly nontrivial; however, due to the simple form of the ϕ_m and ψ_m, we can make quite a bit of progress. For this, we state the following result whose proof is an elementary exercise in calculus.

Lemma 4 *Consider the functions*

$$\phi_m(x) = \cos(mx), \quad \psi_m(x) = \cos(mx),$$

if we define

$$C_{jm} := \left\langle \phi_j, \psi_m \right\rangle, \quad C_{j\ell m} := \left\langle \phi_j\phi_\ell, \psi_m \right\rangle,$$

the following are true:

$$C_{11} = C_{22} = \pi, \quad C_{12} = C_{21} = 0,$$

and

$$C_{112} = C_{121} = C_{211} = \frac{\pi}{2},$$
$$C_{111} = C_{221} = C_{212} = C_{122} = C_{222} = 0.$$

With this we can simplify the previous to the following.

$$\pi s_1 + 2(\pi/2)\beta = 0,$$
$$\pi s_1 \beta + (\pi/2) = 0.$$

The first equation can be solved to give $s_1 = -\beta$, which can be inserted into the second equation to give

$$\beta = \pm\frac{1}{\sqrt{2}} \quad \Longrightarrow \quad s_1 = \mp\frac{1}{\sqrt{2}}.$$

Please compare with the results found in [56].

6.2 Analyticity

We are now in a position to establish the existence and analyticity of branches of Wilton ripple solutions. To estimate solutions of (16), we require the following theorem.

Lemma 5 *For any real $\sigma \geq 0$, given $Q \in H^\sigma$ there exists a unique solution triple $\{F, s, B\}$ of*

$$\mathcal{R}F - sf - (2f + v)B = Q, \quad \langle F, \phi_1 \rangle = \langle F, \phi_2 \rangle = 0, \tag{18}$$

where

$$f = \phi_1 + \beta\phi_2, \quad B = b\phi_2, \quad v \in \mathbf{R},$$

satisfying

$$\max\left\{\|F\|_{H^{\sigma+\tau}}, |s|, \|B\|_{H^{\sigma+\tau}}\right\} \leq C_e \|Q\|_{H^\sigma},$$

for some universal constant $C_e > 0$.

Proof We begin by ensuring solvability which we guarantee by choosing $\{s, b\}$ appropriately. Taking the inner product of (18) with ψ_j, $j = 1, 2$, yields

$$(C_{1j} + \beta C_{2j})s + (2C_{12j} + 2\beta C_{22j} + vC_{2j})b = -\left\langle Q, \psi_j \right\rangle.$$

These lead to the linear system

$$\begin{pmatrix} C_{11} + \beta C_{21} & 2C_{121} + 2\beta C_{221} + vC_{21} \\ C_{12} + \beta C_{22} & 2C_{122} + 2\beta C_{222} + vC_{22} \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} = -\begin{pmatrix} \langle Q, \psi_1 \rangle \\ \langle Q, \psi_2 \rangle \end{pmatrix}.$$

From Lemma 4, this simplifies to

$$\begin{pmatrix} \pi & \pi \\ \pi\beta & \pi v \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} = -\begin{pmatrix} \langle Q, \psi_1 \rangle \\ \langle Q, \psi_2 \rangle \end{pmatrix}.$$

Using the Cauchy–Schwartz inequality, we find

$$\max\{|s|, |b|\} \leq \max_{j=1,2} \left|\left\langle Q, \psi_j \right\rangle\right| \leq \frac{1}{\sqrt{\pi}} \|Q\|_{H^\sigma} .$$

With these choices of $\{s, b\}$ we readily discover that

$$\mathcal{R}F = Q + sf + (2f + v)B \in \operatorname{ran}(\mathcal{R}),$$

so that the estimate on F is clearly true from the mapping properties of $\mathcal{R}$, (3) and (4). $\square$

We now prove our crucial recursive estimate with an eye towards our inductive proof. As we will see, while the recursion for the case of Stokes waves begins at order $n = 2$, for Wilton ripples we must wait until $n = 3$. Thus, we assume our estimates for $3 \leq n \leq N - 1$ and examine $\mathcal{Q}_N$ for $N \geq 4$. We point out that this relegates the quantities (F_2, s_1) "before" the scope of our recursion and it is important that they not be confused with the terms (F_n, s_{n-1}, B_{n-1}), $n \geq 3$, which are estimated by our Theorem.

Lemma 6 *For any real* $\sigma > 1/2$*, if*

$$\|F_n\|_{H^{\sigma+\tau}} \leq C_F \frac{D^{n-3}}{(n+1)^2}, \qquad 3 \leq n \leq N-1,$$

$$|s_{n-1}| \leq C_s \frac{D^{n-3}}{(n+1)^2}, \qquad 3 \leq n \leq N-1,$$

$$\|B_{n-1}\|_{H^{\sigma+\tau}} \leq C_B \frac{D^{n-3}}{(n+1)^2}, \qquad 3 \leq n \leq N-1,$$

for $N \geq 4$*, then we have*

$$\|\mathcal{Q}_N\|_{H^\sigma} \leq \max\{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\} \tilde{C} \frac{D^{N-4}}{(N+1)^2}, \quad N \geq 4,$$

c.f. (17), for some $\tilde{C} > 0$.

Proof We begin with the case $N = 4$ as this requires special attention. We recall that

$$\begin{aligned}\mathcal{Q}_4 &= (2f_1 + s_1)F_3 + (s_2 + B_2 + F_2)(B_2 + F_2) \\ &= 2f_1F_3 + s_1F_3 + s_2B_2 + B_2B_2 + F_2B_2 + s_2F_2 + B_2F_2 + F_2F_2,\end{aligned}$$

and estimate (requiring $\sigma > 1/2$)

$$\begin{aligned}\|\mathcal{Q}_4\|_{H^\sigma} \le\ & 2M\,\|f_1\|_{H^\sigma}\,\|F_3\|_{H^\sigma} + |s_1|\,\|F_3\|_{H^\sigma} + |s_2|\,\|B_2\|_{H^\sigma}\\ & + M\,\|B_2\|_{H^\sigma}^2 + M\,\|F_2\|_{H^\sigma}\,\|B_2\|_{H^\sigma} + |s_2|\,\|F_2\|_{H^\sigma}\\ & + M\,\|B_2\|_{H^\sigma}\,\|F_2\|_{H^\sigma} + M\,\|F_2\|_{H^\sigma}^2\,.\end{aligned}$$

We can use our recursive estimate on the terms s_2, B_2, F_3 leading to the estimate

$$\begin{aligned}\|\mathcal{Q}_4\|_{H^\sigma} \le\ & 2M\,\|f_1\|_{H^\sigma}\,\frac{C_F}{4^2} + |s_1|\,\frac{C_F}{4^2} + \frac{C_s}{4^2}\frac{C_B}{4^2} + M\frac{C_B^2}{4^2 4^2}\\ & + M\,\|F_2\|_{H^\sigma}\,\frac{C_B}{4^2} + \frac{C_s}{4^2}\,\|F_2\|_{H^\sigma} + M\frac{C_B}{4^2}\,\|F_2\|_{H^\sigma} + M\,\|F_2\|_{H^\sigma}^2\\ \le\ & \max\{C_F, C_B, \|F_2\|_{H^\sigma}\}\,\tilde{C},\end{aligned}$$

provided that

$$\begin{aligned}\tilde{C} \ge\ & 2M\,\|f_1\|_{H^\sigma}\,\frac{1}{4^2} + |s_1|\,\frac{1}{4^2} + \frac{C_s}{4^2}\frac{1}{4^2} + M\frac{C_B}{4^2 4^2}\\ & + M\,\|F_2\|_{H^\sigma}\,\frac{1}{4^2} + \frac{C_s}{4^2} + M\frac{1}{4^2}\,\|F_2\|_{H^\sigma} + M\,\|F_2\|_{H^\sigma}\,.\end{aligned}$$

We now examine the case $N \ge 5$ and recall that

$$\begin{aligned}\mathcal{Q}_N =\ & (2f_1 + s_1)\,F_{N-1} + \sum_{\ell=2}^{N-2} s_{n-\ell}B_\ell + \sum_{\ell=2}^{N-2} B_{n-\ell}B_\ell\\ & + \sum_{\ell=2}^{N-2} F_{n-\ell}B_\ell + \sum_{\ell=2}^{N-2} s_{n-\ell}F_\ell + \sum_{\ell=2}^{N-2} B_{n-\ell}F_\ell + \sum_{\ell=2}^{N-2} F_{n-\ell}F_\ell.\end{aligned}$$

The only terms which cause any issue are those involving F_2 and so we separate these out

$$\begin{aligned}\mathcal{Q}_N =\ & (2f_1 + s_1)\,F_{N-1} + \sum_{\ell=2}^{N-2} s_{n-\ell}B_\ell + \sum_{\ell=2}^{N-2} B_{n-\ell}B_\ell\\ & + \sum_{\ell=2}^{N-3} F_{n-\ell}B_\ell + \sum_{\ell=3}^{N-2} s_{n-\ell}F_\ell + \sum_{\ell=3}^{N-2} B_{n-\ell}F_\ell + \sum_{\ell=3}^{N-3} F_{n-\ell}F_\ell\\ & + (s_{N-2} + 2B_{N-2} + 2F_{N-2})\,F_2.\end{aligned}$$

Now we estimate

$$
\begin{aligned}
\|\mathcal{Q}_N\|_{H^\sigma} \le{}& \left(2M\,\|f_1\|_{H^\sigma} + |s_1|\right) C_F \frac{D^{N-4}}{(N-1+1)^2} \\
&+ \sum_{\ell=2}^{N-2} C_s \frac{D^{N-\ell-2}}{(N-\ell+2)^2} C_B \frac{D^{\ell-2}}{(\ell+2)^2} \\
&+ \sum_{\ell=2}^{N-2} C_B \frac{D^{N-\ell-2}}{(N-\ell+2)^2} C_B \frac{D^{\ell-2}}{(\ell+2)^2} \\
&+ \sum_{\ell=2}^{N-3} C_F \frac{D^{N-\ell-3}}{(N-\ell+1)^2} C_B \frac{D^{\ell-2}}{(\ell+2)^2} \\
&+ \sum_{\ell=3}^{N-2} C_s \frac{D^{N-\ell-2}}{(N-\ell+2)^2} C_F \frac{D^{\ell-3}}{(\ell+1)^2} \\
&+ \sum_{\ell=3}^{N-2} C_B \frac{D^{N-\ell-2}}{(N-\ell+2)^2} C_F \frac{D^{\ell-3}}{(\ell+1)^2} \\
&+ \sum_{\ell=3}^{N-3} C_F \frac{D^{N-\ell-3}}{(N-\ell+1)^2} C_F \frac{D^{\ell-3}}{(\ell+1)^2} \\
&+ \left(C_s \frac{D^{N-4}}{(N-2+2)^2} + 2C_B \frac{D^{N-4}}{(N-2+2)^2} \right. \\
&\left. + 2C_F \frac{D^{N-5}}{(N-2+1)^2} \right) \|F_2\|_{H^\sigma} \,.
\end{aligned}
$$

We continue

$$
\begin{aligned}
\|\mathcal{Q}_N\|_{H^\sigma} \le{}& C' \left(2M\,\|f_1\|_{H^\sigma} + |s_1|\right) C_F \frac{D^{N-4}}{(N+1)^2} \\
&+ C_B S C_s \frac{D^{N-4}}{(N+1)^2} + C_B S C_B \frac{D^{N-4}}{(N+1)^2} \\
&+ C_F S C_B \frac{D^{N-5}}{(N+1)^2} + C_s S C_F \frac{D^{N-5}}{(N+1)^2} \\
&+ C_B S C_F \frac{D^{N-5}}{(N+1)^2} + C_F S C_F \frac{D^{N-6}}{(N+1)^2} \\
&+ C' \left(C_s \frac{D^{N-4}}{(N+1)^2} + 2C_B \frac{D^{N-4}}{(N+1)^2} + 2C_F \frac{D^{N-5}}{(N+1)^2} \right) \|F_2\|_{H^\sigma} \,,
\end{aligned}
$$

for some $C' > 0$ which bounds terms like $(N+1)^2/N^2$, and we are done provided

$$\tilde{C} \geq C' \left(2M \|f_1\|_{H^\sigma} + |s_1|\right) + S(2C_s + 3C_B + C_F) + C'(C_s + 2C_B + 2C_F).$$

$\square$

At last we can state and prove our main theorem on existence and parametric analyticity of Wilton ripples.

Theorem 2 *For any real $\sigma > 1/2$*

$$\|F_n\|_{H^{\sigma+\tau}} \leq C_F \frac{D^{n-3}}{(n+1)^2}, \qquad n \geq 3, \tag{19a}$$

$$|s_{n-1}| \leq C_s \frac{D^{n-3}}{(n+1)^2}, \qquad n \geq 3, \tag{19b}$$

$$\|B_{n-1}\|_{H^{\sigma+\tau}} \leq C_B \frac{D^{n-3}}{(n+1)^2}, \qquad n \geq 3, \tag{19c}$$

for some $C_F, C_s, C_B > 0$.

Proof We work by induction in n. The base case $n = 3$ is straightforward, and we now suppose that (19) is true for all $n < N$ for $N \geq 4$. Using Lemma 5 we estimate

$$\max\left\{\|F_N\|_{H^{\sigma+\tau}}, |s_{N-1}|, \|B_{N-1}\|_{H^{\sigma+\tau}}\right\} \leq C_e \|Q_N\|_{H^\sigma},$$

and from Lemma 6 we find

$$\begin{aligned} &\max\left\{\|F_N\|_{H^{\sigma+\tau}}, |s_{N-1}|, \|B_{N-1}\|_{H^{\sigma+\tau}}\right\} \\ &\quad \leq C_e \max\left\{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\right\} \tilde{C} \frac{D^{N-4}}{(N+1)^2}. \end{aligned}$$

The estimate for F_N follows if

$$C_e \max\left\{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\right\} \tilde{C} \frac{D^{N-4}}{(N+1)^2} \leq C_F \frac{D^{N-3}}{(N+1)^2},$$

which is possible by choosing

$$D > C_e \tilde{C} \max\left\{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\right\} / C_F.$$

The estimate for s_{N-1} proceeds provided

$$C_e \max\left\{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\right\} \tilde{C} \frac{D^{N-4}}{(N+1)^2} \leq C_s \frac{D^{N-3}}{(N+1)^2},$$

which is possible by choosing

$$D > C_e \tilde{C} \max \{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\} / C_s.$$

Finally, the bound on B_{N-1} is verified if

$$C_e \max \{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\} \tilde{C} \frac{D^{N-4}}{(N+1)^2} \leq C_B \frac{D^{N-3}}{(N+1)^2},$$

which is possible by choosing

$$D > C_e \tilde{C} \max \{C_F, C_B, \|F_2\|_{H^{\sigma+\tau}}\} / C_B,$$

and we are done. □

Once again, we state the implication of this theorem as a corollary for additional emphasis.

Corollary 2 *The functions $s(\varepsilon)$, $B(x;\varepsilon)$, and $F(x;\varepsilon)$ of (14) are real analytic functions of ε in a sufficiently small neighborhood of the origin in the complex plane. More precisely, for any $0 \leq \rho < 1$, the Taylor series expansions*

$$s(\varepsilon) = \sum_{n=1}^{\infty} s_n \varepsilon^n, \quad B(x;\varepsilon) = \sum_{n=2}^{\infty} B_n(x) \varepsilon^n, \quad F(x;\varepsilon) = \sum_{n=2}^{\infty} F_n(x) \varepsilon^n,$$

c.f. (9), converge for all ε such that $D\varepsilon < \rho$, i.e., $\varepsilon < \rho/D$.

7 Conclusions

In this contribution, we have examined the existence and parametric analyticity of branches of Wilton ripples which bifurcate from the trivial branch of solutions for a family of weakly nonlinear wave equations of the form (1). Our method of proof is quite direct (the "Method of Majorants" [18]) and justifies a spectrally convergent high-order perturbation scheme for the numerical simulation of these traveling waves [3]. This is in contrast to the proof of Reeder and Shinbrot [41,42] whose iterative approach to the full water wave equations would not justify our numerical scheme. We point out that our main result, Theorem 2, and its corollary, Corollary 2, can be generalized in a number of ways which should be of interest to the community. While it is required that the linear operator $\mathcal{L}$ possess a two-dimensional null space, otherwise it can be quite general, even permitting non-polynomial forms (e.g., those of the Benjamin, Whitham, and Akers–Milewski equations; see (1b)). Regarding the nonlinearity, $N(u)$, provided that it is an *analytic* function, our entire proof can be repeated with minor, though tedious, modifications. An immediate goal of the authors is to extend the present approach to the full water wave problem which, while quite intricate and complicated, now appears to be in view.

Dedication

The work of Walter Craig has been deeply influential in a number of fields of Mathematics and Physics scattered about Analysis, Ordinary and Partial Differential Equations, Fluid Mechanics, Mathematical Physics, and Numerical Analysis. One can consult the volume of papers summarizing talks given at his 60th birthday conference at the Fields Institute for ample evidence of this [19]. However, nowhere has his work been more important than in the problem of modeling the free-surface evolution of surface water waves under the influence of gravity and surface tension. In particular, his view of this system as a Hamiltonian system (following the ground-breaking contribution of Zakharov [59]) has been profoundly important in the field and will likely guide researchers for many decades to come.

Walter was the Ph.D. supervisor of one of us (D.P.N.) [36] and an important mentor to the other (B.F.A.). He was an insightful and generous teacher to both of us, and we valued his opinion highly. It was with great sadness that we learned of his death in January of 2019 and he is greatly missed by the large and diverse group of friends and collaborators that he gathered over the years.

Walter was both a leader and friend in the early stages of my (B.F.A.) career. I visited McMaster in the months after receiving my Ph.D. at Walter's invitation, and found myself invited not only to his seminar but also his home. In later years, he found time to join me, and fellow young researchers, for many a conference lunch, or dinner, or even a bit of punting. In each instance, he brought a contagious air of excitement and style that made the occasion more than it might have been; we were not just working, we were part of something. Perhaps an adventure.

Before I (D.P.N.) visited Brown University in the Spring of 1993 to investigate the generous Graduate Assistantship which I had been offered by the Division of Applied Mathematics, I discussed the department with my undergraduate advisor (Paul Newton then at Illinois) who is also an alumnus. We examined every faculty member listed on the Peterson's Guide profile and I dutifully recorded every pearl of wisdom. He then told me that I should definitely make a special visit over to the Mathematics Department to talk with someone he knew from his days as a Post-Doctoral Fellow at Stanford: Walter Craig. During that meeting Walter handed me a preprint of his recent paper with Catherine Sulem, "Numerical Simulation of Gravity Waves" [13]. Reading that paper constituted a seminal moment in my career, and it makes me a little sad that never again will I be able to wander over to Kassar House to ask Walter for a hint (or answer!) to my latest mathematical puzzle.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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