SURFACTANT INFLUENCE ON WATER WAVE PACKETS

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Abstract. The influence of surfactant on water wave packets is investigated. An envelope equation for a slowly varying wave packet in the potential flow equations with variable Bond number is derived. The properties of this equation depend on the relative phases of the wave packet and the distribution of surface tension. We observe that small variations in the Bond number may change the focusing nature of the envelope equation from that of the constant Bond number problem. Variations in Bond number can thus suppress, or incite, the Benjamin-Feir instability. The existence of envelope solitary waves depends in a similar way on the Bond number variation. The envelope equation is also derived in a larger class of models.

1. Introduction. The existence and stability of traveling water waves has been studied for over a century; Stokes derived the speed-amplitude relationship of periodic waves in 1845 [1]. Depending on the depth of the water and the relative strength of the surface tension, periodic traveling waves are subject to modulational instabilities - the most famous of which is the Benjamin-Feir instability [2]. In this work we discuss the modulational stability of periodic waves as well as the existence of wave packet type solitary waves by deriving an approximate envelope equation. Envelope equations have been derived for weakly nonlinear wave packets in deep water as well as on more arbitrary fluid domains and in many other physical contexts [3].

Experimental evidence suggests that surfactant based damping [4] can stabilize the Benjamin-Feir instability. Mathematically, the Benjamin-Feir instability can be modeled in the context of the cubic nonlinear Schrödinger equation (NLS), where the the addition of a small linear damping term has been shown to stabilize the instability [4]. This result has spurred much recent interest in modeling the physical cause of this damping both in the water wave equations and in NLS [5]. The effect of small viscosity has been modeled as a damping term in both the water wave equations and the Nonlinear Schrödinger equation [6, 7]. In the experiments of Henderson [4], the damping is attributed to surfactant deposition on the free surface - periodic waves on "clean" surfaces are Benjamin-Feir unstable, while "dirty" surfaces support modulationaly stable periodic waves.

That the presence of surfactant can damp water waves is well known - famous observations of this phenomenon were made by Benjamin Franklin in 1774 [8] and Reynolds in 1880 [9]. Surfactants are known to influence surface rheology; there are many experimental and theoretical works investigating the linear behavior of water waves in the presence of surfactant [10, 11]. Weakly nonlinear theories have been developed which model surfactants via linear damping terms in cubic envelope equations [12, 13]. In this work we derive an envelope equation including nonlinear quartet interactions from a simple model for water waves in the presence of surfactant - the potential flow equations with small spatiotemporal variations in the Bond number. This envelope equation contains no linear damping, and reduces to the cubic nonlinear Schrödinger (NLS) or complex cubic Ginzburg-Landau equations for special distributions of surface tension. We observe that simple coupling of the Bond number to the wave may change the focusing nature of the envelope equation from that of the constant Bond number case; waves which are Benjamin-Feir unstable may become stable in the presence of variable surface tension without any linear damping.

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Envelope equations can also be used to predict the existence of wave packet type solitary waves - by predicting the asymptotics of a wave's shape. Wave packet type solitary water waves, or capillary-gravity solitary waves, are well known to be predicted by such asymptotic arguments [14, 15]. These traveling solitary water waves are supported near frequencies which are both local extrema of the phase speed (so that the envelope and the carrier wave travel at the same speed) and have focusing envelope equations (so that the envelope can be localized). Because of the latter requirement, the existence of these waves can be predicted with the envelope equation derived here.

2. Derivation. In this section we derive an envelope equation for weakly nonlinear traveling wavepackets in the potential flow equations with variable surface tension. To begin, recall the potential flow equations for a body of water undergoing an irrotational motion, which acts under the forces of gravity and surface tension

$$\phi_{xx} + \phi_{zz} = 0, \qquad -H < z < \epsilon\eta, \tag{2.1a}$$
$$\phi_z = 0, \qquad z = -H, \tag{2.1b}$$
$$\eta_t + \epsilon\eta_x\phi_x = \phi_z, \qquad z = \epsilon\eta, \tag{2.1c}$$
$$\phi_t + \frac{\epsilon}{2}\phi_x^2 + \frac{\epsilon}{2}\phi_z^2 + \eta - \sigma\frac{\eta_{xx}}{(1+|\nabla\eta|^2)^{3/2}} = 0, \qquad z = \epsilon\eta. \tag{2.1d}$$

This equation has been non-dimensionalized as in [16], so that σ is the Bond number, $\sigma = \frac{\gamma}{gL}$ measuring the relative strength of gravity and capillary forces (sometimes defined as the reciprocal) and $\epsilon = \frac{a}{L}$ is the wave slope measuring the relative size of the nonlinearity - a is a characteristic scale for the free surface amplitude, L is the spatial period of the wave. As a simple model for the presence of surfactants, we will take σ to have small variations on the scale of the traveling wave, $\sigma = \sigma_0(1+\epsilon\tilde{\sigma}(x,t))$. Admittedly, the physics of surfactants is much more complicated - they may add elasticity to the surface, diffuse into the fluid creating boundary layers, damp the flow at contact lines, and much more [17]. An accurate model for surfactants should include information about the fluid rheology as well as the motion, including solubility and transport, of the surfactant [10, 11, 18]. Rather than modeling a particular surfactant, assuming a constitutive law to couple surfactant concentration to surface tension like the Boussinesq-Scriven law [19], we consider here only the simplest possible model, that the Bond number depends on space and time, as in [20]. Because viscous stress, and its effect on surfactant evolution, is neglected the linear damping term derived in Miles [12], and again by Joo, Messiter & Schultz [13], does not appear in the envelope equation. Instead, a significantly simpler calculation reveals that variations in the Bond number can alter the nonlinear structure of the envelope equation. An envelope equation which was focusing for constant Bond number can become defocusing when the Bond number is allowed to vary. The envelope equation may be derived directly from (2.1), we simplify the calculation using a cubic approximation as an intermediary.

A cubic model equation is derived from equation (2.1) in three steps, as in [16]. First the free surface boundary conditions are Taylor expanded about the mean $z = 0$. Next, Laplace's equation in a strip is solved for the z-dependence, essentially applying the Dirichelet-to-Neumann map. Finally, the equations are truncated at cubic order and the potential is eliminated in favor of the free surface displacement. The resulting cubic approximation to the potential flow equations is

$$\eta_{tt} + \mathcal{L}S\eta + \epsilon Q[\eta] + \epsilon^2 C[\eta] + \epsilon\sigma_0 F[\eta,\tilde{\sigma}] + \epsilon^2\sigma_0 G[\eta,\tilde{\sigma}] = 0 \tag{2.2}$$

with

$$
\begin{aligned}
Q[\eta] &= \mathcal{L}\left(\frac{1}{2}\eta_t^2 + \frac{1}{2}\mathcal{H}\eta_t^2 - \eta\Omega^2\eta\right) - \left(\eta_t\mathcal{H}\eta_t + \eta S\eta_x\right)_x,\\
C[\eta] &= \left(\eta\eta_t\eta_{xt} - \frac{1}{2}\eta^2\mathcal{L}S\eta_x + \eta_t\mathcal{H}(\eta\mathcal{H}\eta_t)_x - \eta\left(\frac{1}{2}\eta_t^2 + \frac{1}{2}\mathcal{H}\eta_t^2 - \eta\Omega^2\eta\right)_x\right)_x\\
&\quad + \mathcal{L}\left(\eta_t\mathcal{L}\mathcal{H}(\eta\mathcal{H}\eta_t) - \mathcal{H}\eta_t\mathcal{L}(\eta\mathcal{H}\eta_t) - \eta\mathcal{L}\left(\frac{1}{2}\eta_t^2 + \frac{1}{2}\mathcal{H}\eta_t^2 - \eta\Omega^2\eta\right)\right)\\
&\quad + \mathcal{L}\left(\frac{1}{2}(\sigma_0\eta_x)_x^3 + \eta(\mathcal{H}\eta_t\mathcal{L}\mathcal{H}\eta_t + \eta_t\mathcal{L}\eta_t) + \frac{1}{2}\eta^2 S\eta_{xx}\right)\\
F[\eta,\tilde{\sigma}] &= -\mathcal{L}(\tilde{\sigma}\eta_{xx})\\
G[\eta,\tilde{\sigma}] &= \mathcal{L}(\eta\mathcal{L}(\tilde{\sigma}\eta_{xx})) + (\eta(\tilde{\sigma}\eta_{xx})_x)_x
\end{aligned}
$$

which reduces to the model of [16] if $\tilde{\sigma} = 0$. The operator $\mathcal{L}$ is that which is induced by z-derivatives on the trace of the potential, at $z = 0$, in equation (2.1), whose Fourier symbol is $\widehat{\mathcal{L}\eta} = |k|\tanh(|k|H)\hat{\eta}$. The operator $\mathcal{H}$ is the Hilbert transform, $\widehat{\mathcal{H}\eta} = -i\text{sign}(k)\hat{\eta}$ and $S = (1 - \sigma_0\partial_x^2)$. Because the model (2.2) is approximates the potential flow equation (2.1) including nonlinear contributions to cubic order, the quartet equation derived here will be the same as that derived from the potential flow equations (2.1) - quartets are an cubic phenomenon.

The classic ansatz for deriving a quartet envelope equation assumes a carrier wave train with an envelope which varies slowly in space and time

$$
\eta = A(\xi,\tau)e^{i\theta} + * + \epsilon\eta_1 + \epsilon^2\eta_2 + ..., \tag{2.3}
$$

with $\xi = \epsilon(x - c_g t)$, $\tau = \epsilon^2 t$ and $\theta = x - c_p t$. Here and throughout, $*$ stands for complex conjugate. For simplicity, we focus on the deep water limit, $H \to \infty$, where in the constant surface tension case $\tilde{\sigma} = 0$, we recover the $O(\epsilon^2)$ solvability condition

$$
iA_\tau + \lambda A_{\xi\xi} + \chi_0|A|^2 A = 0 \tag{2.4}
$$

This nonlinear Schrödinger equation (2.4) can be used to predict the stability of Stokes' waves, as well as the existence of envelope solitary waves, via the relative signs of its coefficients. A Stokes wave, the spatially constant solution to equation (2.4), is modulationally, Benjamin-Feir, unstable if the product $\lambda\chi < 0$. For the potential flow equatons, the coefficients are

$$
\lambda = \frac{3\sigma_0^2 + 6\sigma_0 - 1}{8\sqrt{1+\sigma_0}} \qquad \text{and} \qquad \chi_0 = \frac{\sqrt{1+\sigma_0}}{2}\left(\frac{2\sigma_0^2 + \sigma_0 + 8}{(1+\sigma_0)(2-4\sigma_0)}\right),
$$

which have been computed in a variety of contexts [21, 22]. Stokes wave solutions of equation (2.4) are unstable for $\sigma_0 < \frac{2\sqrt{3}-3}{3}$, where the boundary is due to a root of λ at the minimum of the linear group velocity, and $\sigma_0 > 1/2$, where the boundary is due to the singularity of χ_0 at the triad resonance known as Wilton's ripple. The envelope equation of this work is the analogy of equation (2.4) when there are small changes in the distribution of surface tension in both space and time, $\tilde{\sigma} \neq 0$.

In this work, as a constitutive relation has not been assumed to predict the relationship between surfactant concentration and Bond number, the Bond number variation is instead treated as a known function. To begin, the Bond number variation is decomposed into a series of normal modes, assuming spatial variations which are superharmonic relative to the Stokes wave, each with a known temporal frequency Ω_n,

$$\tilde{\sigma} = \sum_n B_n e^{inx - i\Omega_n t}. \tag{2.5}$$

Substituting this form for $\tilde{\sigma}$ as well as ansatz (2.3) into (2.2) yields a forced linear equation for η_1,

$$\eta_{1,tt} + LS\eta_1 + 2(1+\sigma_0)A^2 e^{2i\theta} + \sigma_0 \sum_n |n+1| A B_n e^{i(n+1)x - i(c_p + \Omega_n)t} + * = 0, \tag{2.6}$$

where $\theta = x - c_p t$. This equation is resonantly forced if either of the harmonics $e^{2i\theta}$ or $e^{i(n+1)x - i(c_p+\Omega_n)t}$ are proportionate to free wave solutions, $e^{inx - i\omega_n t}$, of the linear equation

$$\eta_{tt} + LS\eta = 0.$$

This first forcing term in (2.6), the second harmonic of the Stokes wave, is responsible for a triad resonance if $\omega_2 = \pm 2\omega_1$. Such resonances are called Wilton's ripples and occur at a countable set of Bond numbers [23, 24]. The second forcing term in equation (2.6) causes a triad resonance when $\omega_1 + \Omega_n = \pm\omega_{n+1}$. In either case, the appropriate evolution equation for the wave amplitude, A, would be a triad equation, requiring that the amplitude depends on $T = \epsilon t$, see [3]. Such resonances have been shown to play an important role for water waves with small periodic vorticity or topography [25, 26].

We now focus on the special case $\tilde{\sigma} = \tilde{\sigma}(x - c_p t)$, or $\Omega_n = nc_p$, where the variations in Bond number travel at the phase speed of the wave. Although the relationship between surfactant and Bond number is not modeled here, it is worth noting that one would expect the surfactant particles to be transported with the surface flow, at the Stokes drift $O(\epsilon^2)$ - as well as diffusing into the fluid and being deposed upon it at other rates. On the other hand oscillations in the surfactant concentration may move with at the wave speed without net transport of surfactant - analogous to how the free surface displacement travels without the free surface particles traveling. Of course the existence of such traveling waves in surfactant concentration depends on the model chosen for surfactant dynamics, see Joo et. al. [13] for an example of a model where the Bond number variations move at the phase speed. Proceeding with a traveling $\tilde{\sigma}$, the variation in Bond number is written as a sum of traveling harmonics with slowly varying envelope

$$\tilde{\sigma} = \sum B_n(X,T) e^{in\theta}, \tag{2.7}$$

where $X = \epsilon x$, $T = \epsilon t$ and by definition $B_0 = 0$. Substituting the ansatz (2.3) into equation (2.2), the correction η_1 is solves the following forced linear equation

$$\eta_{1,tt} + LS\eta_1 + 2(1+\sigma_0)A^2 e^{2i\theta} + \sigma_0 \sum |n+1| A B_n e^{i(n+1)\theta} + * = 0 \tag{2.8}$$

If B_2 (or B_{-2}) is nonzero this equation will be resonantly forced, and a solution of the form of equation (2.3) does not exist due to a triad resonance. As triad resonances exist for only a small set of depths and Bond numbers σ_0 [24], and the goal here is to investigate the Benjamin-Feir instability and envelope solitary waves, both based on quartet resonances, we assume that there is not a triad resonance at σ_0.

If the surface tension is not supported at the second harmonic, or the size of the support is sufficiently small, then there are no resonant terms in equation (2.8), and the first correction to the free surface is

$$\eta_1 = \alpha A^2 e^{2i\theta} + \sigma_0 \sum_n \beta_{n+1} A B_n e^{i(n+1)\theta} \tag{2.9}$$

where

$$\alpha = \frac{(1+\sigma_0)}{1-2\sigma_0},$$
$$\beta_n = \frac{|n|}{n^2(1+\sigma_0) - |n|(1+\sigma_0 n^2)},$$

for $|n| > 1$ and defining $\beta_0 = \beta_1 = \beta_{-1} = 0$. For a bounded correction η_2 to exist, one must impose the solvability condition

$$\begin{aligned} iA_\tau + \lambda A_{\xi\xi} + \chi_0 |A|^2 A + \chi_1 \bar{B}_1 A^2 + \chi_2 |A|^2 B_1 + \chi_3 \bar{A}^2 B_3 \\ + \sum \gamma_j B_j B_{2-j} \bar{A} + \sum \rho_j |B_j|^2 A = 0 \end{aligned} \tag{2.10}$$

with

$$\chi_1 = \frac{-2\alpha}{\sqrt{1+\sigma_0}},$$
$$\chi_2 = -\sigma_0 \sqrt{1+\sigma_0}\,\beta_2,$$
$$\chi_3 = -\sigma_0 \frac{\bar{\alpha}(12-4\sigma_0) + 2\beta_2(1+\sigma_0) - 2}{2\sqrt{1+\sigma_0}},$$
$$\gamma_j = -\sigma_0^2 \frac{(j-1)^2(\beta_{j-1} + \beta_{1-j})}{2\sqrt{1+\sigma_0}},$$
$$\rho_j = -\sigma_0^2 \frac{(j+1)^2 \beta_{j+1} + (1-j)^2 \beta_{1-j}}{2\sqrt{1+\sigma_0}}.$$

The properties of equation (2.10) may or may not be similar to that of the NLS equation (2.4) for the constant surface tension case. As an example to illustrate the possibilities, we take $B_j = 0$ for $j \neq 1$ and discuss equation (2.11) for different choices of B_1. If the Bond number varies proportionate to the wave envelope, $B_1 = \mu A$, as is the case in Joo et. al [13], then equation (2.10) becomes the NLS equation

$$iA_\tau + \lambda A_{\xi\xi} + \chi(\mu, \sigma_0)|A|^2 A = 0 \tag{2.11}$$

with

$$\chi(\mu, \sigma_0) = \left(\frac{2\sigma_0^2 + \sigma_0 + 8 - 4\bar{\mu}\sigma_0(1+\sigma_0) - 2\mu\sigma_0(1+\sigma_0) - 8\sigma_0^2|\mu|^2}{4\sqrt{1+\sigma_0}(1-2\sigma_0)} \right).$$

Notice that we have not specified the sign of μ, nor that it is real. Thus the relative phase of B_1, as compared to A, determines the properties of (2.11). It may behave as a focusing or defocusing NLS (for μ real) or a cubic Ginzburg-Landau equation (μ complex). Since the sign of a real χ, relative to λ, determines the stability of a Stoke's wave, we can see that small variations in surface tension may stabilize the Benjamin-Feir instability if they maintain the appropriate phase relative to the carrier

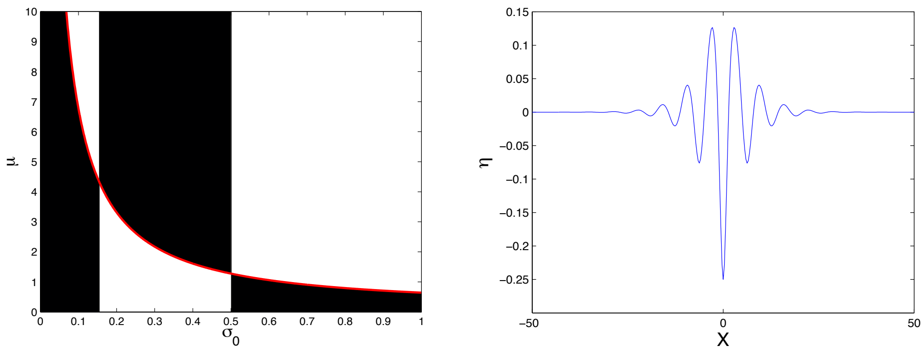

FIG. 2.1. ***Left:*** *The sign of $\lambda\chi$ is depicted as function of the Bond number σ_0 and the real ratio of the first harmonic of the surface tension distribution to that of the traveling wave μ. In the black regions, the NLS is focusing ($\lambda\chi < 0$) and in the white regions it is defocusing ($\lambda\chi > 0$). The sign changes at $\sigma_0 = 1/2$, $\sigma_0 = \frac{2\sqrt{3}-3}{3}$ and at $\mu = -\frac{3(1+\sigma_0)}{8\sigma_0} + \frac{1}{8\sigma_0}\sqrt{73 + 26\sigma_0 + 25\sigma_0^2}$. In the black regions plane, periodic waves are modulationaly unstable - due to the Benjamin-Feir instability.* ***Right:*** *An example of the profile of a gravity-capillary solitary wave solution of equation (2.2) in deep water. This profile exists near $\sigma_0 = 1$, $\mu = 0$ - where the NLS equation is focusing and the speed of the envelope equals the speed of the carrier wave.*

wave. The region of the $\sigma_0\mu$-plane where equation (2.11) is focusing (Benjamin-Feir unstable) is depicted by the black region in the left panel of figure 2.1. Notice that the Benjamin-Feir instability of infinitely long, gravity waves ($\sigma_0 = 0$) persists for all values of μ however, for any finite σ_0 there is a value of μ for which the modulational stability changes.

As a model for the envelope of a water wave, NLS has also been used as a predictive tool for the existence of wave packet like solitary waves [14, 27]. For the water wave problem, wave packet type solitary waves exist near $\sigma_0 = 1$ - with a length scale of about 2cm. An example of a wave packet type, or capillary gravity solitary wave, is in the right panel of figure 2.1. Experimental efforts to observe these waves have seen only transients resembling these waves, where the transient nature is attributed to damping - of both the surfactant and viscous types [28, 29]. With variable surface tension, and appropriate phase difference, complex μ, between the distribution of surface tension and the envelope, equation (2.10) can be dissipative. Moreover, equation (2.10) may be focusing or defocusing, to a degree that depends on the size and phase of the surface tension distribution relative to the wave envelope. The possibility exists here for balancing of the viscous dissipation by an appropriately engineered distribution of surface tension. One could also consider managing the dispersion in the defocusing case, effectively engineering dispersion-managed water wave packets by controlling the surface tension distribution - similar to Feshbach resonance management in Bose-Einstein condensate [30]. The physical relevance, and practicality of engineering, such distributions of surface tension is an open problem.

Of course, the phase of B_j need not be coupled to A. From the perspective of the derivation of the envelope equation, the B_j may have any temporal dynamics in the τ timescale. To find the dynamics of the B_j requires appending to equation (2.1) a model for surfactant transport, see for example [10, 13, 18].

The conclusion for water waves, that small variations in a parameter can change the focusing nature of an envelope equation, is fairly generic. The equivalent of equation (2.11) has been developed here for a family of models which share the form of equation (2.2)

$$\eta_{tt} + \Omega^2\eta + \epsilon Q[\eta,\eta] + \epsilon^2 C[\eta,\eta,\eta] + \epsilon F[\eta,\tilde{\sigma}] + \epsilon^2 G[\eta,\eta,\tilde{\sigma}] = 0 \tag{2.12}$$

where the nonlinear character of the functions Q, C, F, G is denoted by its arguments - in that Q is quadratic in η, C is cubic in η etc, and Ω^2 is a Gallilean invariant linear operator whose Fourier symbol is real. To derive the envelope equation, one must know the action of these terms on harmonics $e^{ik_1(x-ct)}$, so we introduce the notation

$$\hat{F}[k_1,k_2]e^{i(k_1+k_2)(x-ct)} = F(e^{ik_1(x-ct)}, e^{ik_2(x-ct)})$$

with similar definitions for $\hat{Q}[k_1,k_2]$, $\hat{G}[k_1,k_2,k_3]$ and $\hat{C}[k_1,k_2,k_3]$, so that the function $\hat{F}$ plays the role of the Fourier coefficient of F. We will consider models which conserve η_t, e.g. have no mean flow, by assuming

$$\hat{F}[k,-k] = \hat{G}[k,-k] = \hat{C}[k,j,-(k+j)] = \hat{G}[k,j,-(j+k)] = 0.$$

To derive the envelope equation, we will make the generic assumption that the model does not support a triad interaction between the carrier wave frequency and its harmonics, and then write $\tilde{\sigma}$ and η as before as sums of slowly varying envelopes of traveling harmonics - taking the ansatz (2.3) and (2.7). The correction η_1, is then

$$\eta_1 = \frac{\hat{Q}[1,1]}{\Lambda(2)}A^2e^{2i\theta} + \sum_n \frac{\hat{F}[n,1]}{\Lambda(n+1)}AB_n e^{i(n+1)\theta} + *. \tag{2.13}$$

where $\Lambda(k) = \hat{\Omega}^2[k] - k^2\hat{\Omega}[k]^2$. The envelope equation is

$$\begin{aligned} iA_\tau + \lambda A_{\xi\xi} + \chi_0|A|^2A + \chi_1\bar{B}_1A^2 + \chi_2|A|^2B_1 + \chi_3\bar{A}^2B_3 \\ + \sum \gamma_j B_j B_{2-j}\bar{A} + \sum \rho_j |B_j|^2 A = 0 \end{aligned} \tag{2.14}$$

where

$$\lambda = \frac{\Omega''(1)}{2},$$

$$\chi_0 = \frac{-1}{2i\Omega(1)}\left(\frac{\hat{Q}[1,1](\hat{Q}[2,-1]+\hat{Q}[-1,2])}{\Lambda(2)} + \hat{C}[1,1,-1] + \hat{C}[1,-1,1] + \hat{C}[-1,1,1]\right),$$

$$\chi_1 = \frac{-1}{2i\Omega(1)}\left(\frac{\hat{Q}[1,1]\hat{F}[2,-1]}{\Lambda(2)} + \hat{G}[1,1,-1]\right),$$

$$\chi_2 = \frac{-1}{2i\Omega(1)}\left(\frac{\hat{F}[1,1](\hat{Q}[2,-1]+\hat{Q}[-1,2])}{\Lambda(2)} + \hat{G}[1,-1,1] + \hat{G}[-1,1,1]\right),$$

$$\chi_3 = \frac{-1}{2i\Omega(1)}\left(\frac{\hat{Q}[-1,-1]\hat{F}[-2,3]}{\Lambda(-2)} + \frac{\hat{F}[-1,3](\hat{Q}[2,-1]+\hat{Q}[-1,2])}{\Lambda(2)} + \hat{G}[-1,-1,3]\right),$$

$$\gamma_j = \frac{-1}{2i\Omega(1)}\left(\frac{\hat{F}[-1,j+2]\hat{F}[j+1,-j]}{\Lambda(j+1)} + \frac{\hat{F}[-1,j]\hat{F}[j-1,j+2]}{\Lambda(j-1)}\right),$$

$$\rho_j = \frac{-1}{2i\Omega(1)}\left(\frac{\hat{F}[1,j]\hat{F}[j+1,-j]}{\Lambda(j+1)} + \frac{\hat{F}[1,-j]\hat{F}[1-j,j]}{\Lambda(1-j)}\right).$$

The key feature to note in the general problem is that, excluding models with special symmetry, χ_1 does not equal χ_2. If $B_1 = \mu A$, the associated model equation will be a complex Ginzburg-Landau equation for general phase choices μ. For real μ, equation (2.10) will be an NLS equation which is focusing or defocusing depending on μ.

3. Conclusion. An envelope equation was derived for deep water waves with a variable Bond number, as a simple model for water waves in the presence of surfactant. It is shown that the phase of the harmonics of the Bond number relative to that of a traveling wave effects the focusing/defocusing nature of this envelope equation. Small changes in the value of the Bond number can thus stabilize the Benjamin-Feir instability - by changing envelope equation from focusing to defocusing. The model of a variable Bond number is an extreme simplification of the physics of surfactant, however the the envelope equation which is derived is similar to that of a more general family of models. The results motivate future research in physical models for surfactant as well as water wave experiments with intentionally non-constant surface tension.

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