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ORIGINAL ARTICLE

Wilton Ripples with High-Order Resonances in Weakly Nonlinear Models

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Abstract
Resonant periodic traveling waves (Wilton ripples) are examined asymptotically for a family of weakly nonlinear partial differential equations. Wilton ripple resonances can occur between pairs of wavenumbers, here labeled $k = 1$ and N. Typical studies consider $N = 2$, the triad resonance, but this work examines $N > 2$, denoted "high-order resonance." We present explicit formulas for the coefficients in the asymptotic series expansions and answer previously unaddressed questions including what modes are present at each order in the asymptotic expansion and at what order we can expect a non-zero Wilton ripple. The character of the solutions is presented using the example of the Kawahara equation. Finally, we comment on the factors which are indicative of convergence for the asymptotic expansions, and present an example where the series degenerates in the Benjamin equation.

Mathematics Subject Classification Resonant periodic travelling waves · Wilton ripples · Weakly nonlinear water waves · Asymptotic series solutions · Perturbation series

1 Introduction

This work analyzes resonant Wilton ripple solutions [1] to a family of weakly nonlinear dispersive partial differential equations, based on the method developed in [2]. Wilton ripples are branches of traveling waves which, at small amplitudes, are composed of two co-propagating harmonics [3], see Fig. 1 for an example of a ripple between wavenumbers $k = 1$ and $k = 2$. In the water wave problem, Wilton ripples exist at special values of the surface tension coefficient; these special values are those

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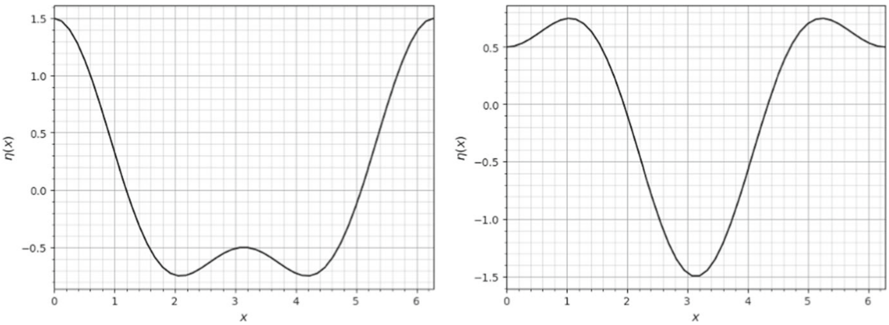


Fig. 1 An example of a Wilton ripples for a resonance between modes $k = 1$ and $k = 2$ given by the expression $\eta(x) = \cos(x) \pm \frac{1}{2}\cos(2x)$ with the positive sign shown on the left plot and the negative sign plotted on the right plot

for which the flat-state linearization has two-dimensional kernel. Wilton ripples have a significant history of study, beginning with asymptotic work [1, 4–7], including numerical computations [2, 8–11], and experiments [12–14]. Wilton ripples have been rigorously shown to exist in the water wave problem [15] and in approximate models [16, 17]. The local and global bifurcation structure has been studied in potential flow, the hydro-elastic problem, and in approximate models [18–21]. Historical studies focus on the triad ripple, between wavenumbers $k = 1$ and $k = 2$ (see Fig. 1), as they require a lower order expansion in the small parameter, meaning the asymptotics are less work to calculate and they also have more physical relevance as these resonances are larger in amplitude; in this work, we develop the general-order asymptotics for higher-order resonances, $k = 1$ and $k = N$ for $N > 2$. These asymptotics are used to write a numerical method, as in [2]. They are also a necessary building block for a future extension of the existence and analyticity proofs in [16] to higher-order resonances.

In constructing the asymptotic solutions, we must consider the spectrum of the operator admitting traveling wave solutions as done in [22, 23]. In particular, the resonant waves we consider here are reminiscent of the form of instabilities examined in [24–26] for an operator perturbed about a traveling wave solution near eigenvalue collisions. Here, we present results on Wilton ripples where the resonant frequencies have large frequency differences and are restricted to integer values (therefore a period of 2π) whereas [11, 27–29] consider perturbations of any period. It should be noted that while we chose to fix the period and expand in terms of wave amplitude, there are other works that do an expansion in terms of period [20] or even consider quasi-periodic waves [30].

The family of partial derivative equations (PDEs) considered is

$$u_t - \mathcal{L}u_x + (u^2)_x = 0, \quad \text{where } x \in [0, 2\pi] \tag{1}$$

as in [2, 17]. Moving to a traveling frame,

$$u(x, t) \to f(x + ct),$$

Table 1 Fourier symbols $\widehat{\mathcal{L}}(k)$, resonant coefficient α and linear speed c_0 for the models represented by (1) where the linear operator allows waves with co-propagating frequencies

Model	$\widehat{\mathcal{L}}(k)$	α	c_0
Kawahara	$k^2 - \alpha k^4$	$\frac{1}{N^2+1}$	$\frac{N^2}{N^2+1}$
Benjamin	$\lvert k\rvert - \alpha k^2$	$\frac{1}{N+1}$	$\frac{N}{N+1}$
Akers–Milewski	$\frac{1}{\lvert k\rvert} + \alpha\lvert k\rvert$	$\frac{1}{N}$	$\frac{N-1}{N}$
Deep-water Whitham	$\sqrt{\frac{1}{\lvert k\rvert} + \alpha\lvert k\rvert}$	$\frac{1}{N}$	$\sqrt{\frac{N-1}{N}}$
Hydroelastic	$\sqrt{\frac{1}{\lvert k\rvert} - \alpha\lvert k\rvert^3}$	$\frac{1}{N^3+N^2+N}$	$\sqrt{\frac{N^3+N^2+N-1}{N^3+N^2+N}}$

then (1) becomes

$$cf_x - \mathcal{L}f_x + (f^2)_x = 0. \tag{2}$$

This equation can be integrated once with respect to x

$$[c - \mathcal{L}]f + f^2 = 0, \tag{3}$$

which will be the form of the equation studied in this paper. The mean of f in (2) can be absorbed into the definition of c, and is here set to zero, as is any constant of integration in (3).

The operator $\mathcal{L}$ is defined via its Fourier symbol $\widehat{\mathcal{L}}(k)$ which is assumed to be real and multiplicative. This restriction includes many important equations, see examples in Table 1, where $\widehat{\mathcal{L}}$ is given for a few commonly used models of water waves. We believe any linear operator $\widehat{\mathcal{L}}$ that is real (to allow for traveling waves) and non-monotonic (to allow for co-propagating ripples) could be used. In each example, a parameter α is chosen to allow for resonance between frequencies $k = 1$ and $k = N$, where the operator, $[c - \mathcal{L}]$, has a two dimensional kernel. Infinitesimal, linear solutions to (3) are then any element of this kernel, $\text{span}(\cos x, \cos Nx)$. Linear theory also predicts that these waves travel at the phase speed, $c_0 = \widehat{\mathcal{L}}(1) = \widehat{\mathcal{L}}(N)$. Wilton ripples are the nonlinear solutions which bifurcate from the flat state at this phase speed. The values of c_0 and resonant coefficient α where ripples exist in each equation are also given in Table 1.

In this work, we show how to construct an asymptotic series solution for models describing water waves described by the PDEs in (1) assuming a small amplitude, periodic, traveling wave solution in the presence of resonance with parameters shown in Table 1. In Sect. 2, we introduce the method using the specific case for resonance at $N = 3$ in the Kawahara model. In Sect. 3, we generalize our findings for the Kawahara model, and derive recursive formulas for the higher-order coefficients. $N \geq 4$ is considered in Sect. 4, where we consider the first three orders explicitly, then obtain recursive formulas similar to those found in Sect. 3. Additionally, this section includes theorems that give insight to which modes are present at which order of the small amplitude parameter. In Sect. 5, we highlight the presence of a common divisor which appears when solving for the coefficient of the resonant mode at each order, and discuss its impact on the radius of convergence of our asymptotic solutions. Finally, we present our conclusions and future directions in Sect. 6.

2 The Kawahara Equation $N = 3$

To demonstrate the method, we illustrate using the Kawahara equation with $N = 3$. Generalizations to other choices of $\mathcal{L}$ and N are to be discussed in later sections. The Kawahara equation is given by $\widehat{\mathcal{L}}(k) = k^2 - \frac{1}{N^2+1}k^4$, and thus for this specific case,

$$\widehat{\mathcal{L}}(k) = k^2 - \frac{1}{10}k^4,$$
$$c_0 = \widehat{\mathcal{L}}(1) = \frac{9}{10}.$$

We begin by defining $\mathcal{R}[f] \equiv [\mathcal{L} - c_0]f$, and equivalently

$$\widehat{\mathcal{R}}(k) \equiv \widehat{\mathcal{L}}(k) - c_0 = k^2 - \frac{1}{10}k^4 - \frac{9}{10}, \tag{4}$$

such that $\widehat{\mathcal{R}}(1) = \widehat{\mathcal{R}}(3) = 0$. With this definition, (3) may be expressed as

$$\mathcal{R}[f] = (c - c_0)f + f^2. \tag{5}$$

The function f and the speed correction $c - c_0$ are expanded asymptotically as

$$f = \epsilon f_1 + \mathcal{O}(\epsilon^2), \quad c - c_0 \equiv s = \mathcal{O}(\epsilon).$$

As both f and s are first order in ϵ, the right-hand side of (5) is second order in ϵ, and thus

$$\mathcal{R}[f_1] = 0.$$

This requires that f_1 must be contained within the kernel of $\mathcal{R}$, or

$$f_1 = \cos(x) + b_1 \cos(3x). \tag{6}$$

In principle, any linear combination of $\cos(x)$ and $\cos(3x)$ are permitted, though ϵ can be redefined to normalize the coefficient of $\cos(x)$ to one. The modes in the solution will be denoted $\phi_m = \cos(mx)$, with inner product

$$\langle \phi_m, \phi_n \rangle = \int_0^{2\pi} \phi_m(x)\phi_n(x)\mathrm{d}x = \pi\delta_{m,n}, \tag{7}$$

where $\delta_{m,n}$ is the Kronecker delta function.

To obtain the higher-order components of f and s, we further expand f as

$$f(x) = \epsilon f_1 + B(x;\epsilon) + F(x;\epsilon), \tag{8}$$

$$B(x;\epsilon) = \sum_{n=2}^{\infty} \epsilon^n B_n(x) = \sum_{n=2}^{\infty} \epsilon^n b_n \phi_3, \tag{9}$$

$$F(x;\epsilon) = \sum_{n=2}^{\infty} \epsilon^n F_n(x) = \sum_{n=2}^{\infty} \epsilon^n \sum_{\substack{k=2,\\ k\neq 3}}^{\infty} \beta_{n,k}\phi_k, \tag{10}$$

where $B(x;\epsilon)$ contains higher-order components of the resonant mode ϕ_3, and $F(x;\epsilon)$ contains higher-order components of all modes orthogonal to the null space of $\mathcal{R}$. As there are no other components of ϕ_1, the parameter ϵ can be associated with the magnitude of the first mode. We also add the transformation

$$s \equiv c - c_0 = \sum_{n=1}^{\infty} \epsilon^n s_n. \tag{11}$$

In terms of the modes, an angle-addition formula for cosine takes the form,

$$\phi_m\phi_n = \frac{1}{2}\left(\phi_{m-n} + \phi_{m+n}\right). \tag{12}$$

Remark If we let $n = m$ in the above, then we get the usual identity for the square of a cosine

$$\phi_m^2 = \frac{1}{2}\left(\phi_0 + \phi_{2m}\right) = \frac{1}{2}\left(1 + \phi_{2m}\right),$$

where $\phi_0 = \cos(0) = 1$.

Using the expansion in the small parameter given by (8)–(11) in (5), yields

$$\mathcal{R}[f] = f(s + f) \tag{13}$$

$$\begin{aligned}\mathcal{R}[F] &= (\epsilon f_1 + B + F)(s + \epsilon f_1 + B + F)\\ &= \{\epsilon s f_1 + \epsilon^2 f_1^2\} + \{(2\epsilon f_1 + s)(B + F)\} + \left\{(B + F)^2\right\},\end{aligned} \tag{14}$$

where terms of $\mathcal{O}(\epsilon^2)$, $\mathcal{O}(\epsilon^3)$, and $\mathcal{O}(\epsilon^4)$ have been separated by curly brackets. Further expansion of (14) gives

$$\mathcal{R}[F_2] = f_1^2 + s_1 f_1, \tag{15}$$

$$\mathcal{R}[F_n] = s_{n-1} f_1 + (2f_1 + s_1)(B_{n-1} + F_{n-1}) + \sum_{l=2}^{n-2}(s_{n-l} + F_{n-l} + B_{n-l})(F_l + B_l). \tag{16}$$

At $\mathcal{O}(\epsilon^n)$, $n > 2$, the unknowns are the function F_n and the constants, s_{n-1} and b_{n-2}. To determine each of the constants, one may appeal to **solvability conditions**,

$$\langle \mathcal{R}[F_n], \phi_1 \rangle = 0, \tag{17}$$

$$\langle \mathcal{R}[F_n], \phi_3 \rangle = 0. \tag{18}$$

These conditions force the right hand side of (16) to be in the range of the operator $\mathcal{R}$. We note that these solvability conditions are due to the Fredholm alternative [31] and formally should contain the adjoint eigenfunctions. However, due to the fact that the operator is self-adjoint and it can be shown that solutions to these model equations are real and symmetric, instead of considering an expansion in the form of e^{ijx} and e^{-ijx}, we simply use $\phi_j = \cos(jx)$. Once the solvability conditions have been applied, the operator $\mathcal{R}$ may be inverted over its range, giving

$$\beta_{n,j} = \frac{\langle \mathcal{R}[F_n], \phi_j \rangle}{\pi \widehat{\mathcal{R}}(j)}, \tag{19}$$

where $\beta_{n,j}$ represents the coefficient of ϕ_j at $\mathcal{O}(\epsilon^n)$.

The general process for solving at each order is as follows:

1. Determine $\mathcal{R}[F_n]$ using (15) for $n = 2$ or (16) for all other n.
2. Utilize the solvability conditions (17) and (18) to generate two equations which are used to solve for s_{n-1} and b_{n-2}.
3. Substitute the values obtained for s_{n-1} and b_{n-2} into the equation obtained for $\mathcal{R}[F_n]$.
4. Utilize (19) to determine the values of each $\beta_{n,j}$.
5. Proceed to next order.

Step 1 with $n = 2$ can be done using (6) in (15) alongside cosine identity (12), to obtain

$$\mathcal{R}[F_2] = s_1\phi_1 + \left(\frac{1}{2} + b_1\right)\phi_2 + s_1 b_1 \phi_3 + b_1\phi_4 + \frac{1}{2} b_1^2 \phi_6 \tag{20}$$

in the case $N = 3$.

Substituting (20) into our solvability conditions, we arrive at

$$\langle \mathcal{R}[F_2], \phi_1 \rangle = \pi s_1 = 0,$$
$$\langle \mathcal{R}[F_2], \phi_3 \rangle = \pi s_1 b_1 = 0,$$

which is satisfied by $s_1 = 0$, leaving b_1 arbitrary. We substitute this result into (20) to obtain

$$\mathcal{R}[F_2] = \frac{1}{2} + b_1\phi_2 + b_1\phi_4 + \frac{1}{2} b_1^2 \phi_6.$$

The coefficients of the solution F_2 can be written explicitly as,

$$\beta_{2,2} = \frac{1}{\widehat{\mathcal{R}}(2)}\left(\frac{1}{2} + b_1\right) = \frac{1}{3} + \frac{2}{3}b_1, \quad \beta_{2,4} = \frac{b_1}{\widehat{\mathcal{R}}(4)} = -\frac{2}{21}b_1,$$
$$\beta_{2,6} = \frac{b_1^2}{2\widehat{\mathcal{R}}(6)} = -\frac{1}{189}b_1^2,$$

where (4) has been used in the final equalities on each line.

We may now proceed to the next order ($\mathcal{O}(\epsilon^3)$), and we use (16) and the form of F_2 found above to write the solvability conditions (17) and (18) as

$$\langle \mathcal{R}[F_3], \phi_1 \rangle = \pi \left(s_2 + \beta_{2,2} + b_1\beta_{2,2} + b_1\beta_{2,4}\right) = 0,$$
$$\langle \mathcal{R}[F_3], \phi_3 \rangle = \pi \left(s_2 b_1 + \beta_{2,2} + \beta_{2,4} + b_1\beta_{2,6}\right) = 0.$$

Substituting in the formulas for each β gives the system of equations

$$s_2 + \frac{1}{2\widehat{\mathcal{R}}(2)} + \frac{3}{2\widehat{\mathcal{R}}(2)}b_1 + \left(\frac{1}{\widehat{\mathcal{R}}(2)} + \frac{1}{\widehat{\mathcal{R}}(4)}\right)b_1^2 = s_2 + \frac{1}{3} + b_1 + \frac{4}{7}b_1^2 = 0, \tag{21}$$

$$\frac{1}{2\widehat{\mathcal{R}}(2)} + \left(s_2 + \frac{1}{\widehat{\mathcal{R}}(2)} + \frac{1}{\widehat{\mathcal{R}}(4)}\right)b_1 + \left(\frac{1}{2\widehat{\mathcal{R}}(6)}\right)b_1^3 = \frac{1}{3} + \left(s_2 + \frac{4}{7}\right)b_1 - \frac{1}{189}b_1^3 = 0. \tag{22}$$

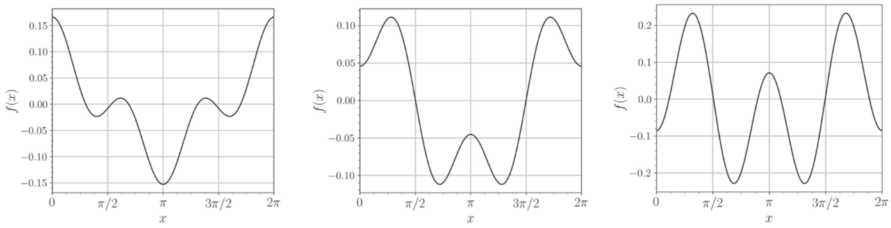
As the above is cubic in b_1, it has three possible solution branches. For the Kawahara equation with $N = 3$, each solution branch is real, and thus each branch is physically significant. The three branches are given by

$$b_1 = 0.59468, \quad s_2 = -1.13,$$
$$b_1 = -0.54488, \quad s_2 = 0.042,$$
$$b_1 = -1.78374, \quad s_2 = -0.368.$$

Each solution pair represents a branch of solutions which bifurcates from the single branch at $c = c_0 = \frac{9}{10}$. The wave profile for each solution is depicted in Fig. 2 for $\epsilon = 0.1$, computed up to $\mathcal{O}(\epsilon^6)$.

Note that the values of $\beta_{2,2}$, $\beta_{2,4}$ and $\beta_{2,6}$ will depend on the choice of b_1, and so too will all other higher-order coefficients. Taking the first solution from above, $b_1 = 0.59468$ and $s_2 = -1.13$, we find the following values for the coefficients in F_3

$$\beta_{3,2} = \frac{b_2}{\widehat{\mathcal{R}}(2)} = \frac{2}{3}b_2, \quad \beta_{3,4} = \frac{b_2}{\widehat{\mathcal{R}}(4)} = -\frac{2}{21}b_2,$$
$$\beta_{3,5} = \frac{b_1\beta_{2,2} + \beta_{2,4} + \beta_{2,6}}{\widehat{\mathcal{R}}(5)} = -9.78 \cdot 10^{-3}, \quad \beta_{3,6} = \frac{b_1 b_2}{\widehat{\mathcal{R}}(6)} = -(6.29 \cdot 10^{-3})b_2,$$
$$\beta_{3,7} = \frac{b_1\beta_{2,4} + \beta_{2,6}}{\widehat{\mathcal{R}}(7)} = 1.85 \cdot 10^{-4}, \quad \beta_{3,9} = \frac{b_1\beta_{2,6}}{\widehat{\mathcal{R}}(9)} = 1.93 \cdot 10^{-6}.$$



(a) $b_1 = 0.59468$ (b) $b_1 = -0.54488$ (c) $b_1 = -1.78374$

Fig. 2 A solution with amplitude $\epsilon = 0.1$ from each of the three branches of solutions for Wilton ripples with $N = 3$

Note that the coefficients of b_2 in $\beta_{3,2}$ and $\beta_{3,4}$ are the same as the coefficients of b_1 in $\beta_{2,2}$ and $\beta_{2,4}$. At this point, we could proceed to $\mathcal{O}(\epsilon^4)$ and beyond; however, this simply involves repeating the steps which have been shown for $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$. As such, we seek to generalize our findings in the next section.

3 General Solution for Resonance at $N = 3$

For other choices of $\mathcal{L}$ with $N = 3$, the only change from the Kawahara version is the form of $\mathcal{R}$. Therefore, for a general $\mathcal{L}$ with $N = 3$, we find $s_1 = 0$,

$$\beta_{2,2} = \frac{1}{\widehat{\mathcal{R}}(2)}\left(\frac{1}{2} + b_1\right), \quad \beta_{2,4} = \frac{b_1}{\widehat{\mathcal{R}}(4)}, \quad \beta_{2,6} = \frac{b_1^2}{2\widehat{\mathcal{R}}(6)},$$

and

$$s_2 + \frac{1}{2\widehat{\mathcal{R}}(2)} + \frac{3}{2\widehat{\mathcal{R}}(2)} b_1 + \left(\frac{1}{\widehat{\mathcal{R}}(2)} + \frac{1}{\widehat{\mathcal{R}}(4)}\right) b_1^2 = 0, \tag{23}$$

$$\frac{1}{2\widehat{\mathcal{R}}(2)} + \left(s_2 + \frac{1}{\widehat{\mathcal{R}}(2)} + \frac{1}{\widehat{\mathcal{R}}(4)}\right) b_1 + \left(\frac{1}{2\widehat{\mathcal{R}}(6)}\right) b_1^3 = 0. \tag{24}$$

This system can be reduced to a single cubic for the parameter b_1, thus three solutions are guaranteed. The exact values of these solutions depend on the choice of $\widehat{\mathcal{L}}$. Of course, not all those solutions need be real, but for all models in Table 1, there are three real solutions. The rest of the asymptotics will be developed in terms of s_2 and b_1. The coefficients of the modes in F_3 can then be calculated as in Sect. 2,

$$\beta_{3,2} = \frac{b_2}{\widehat{\mathcal{R}}(2)}, \quad \beta_{3,4} = \frac{b_2}{\widehat{\mathcal{R}}(4)}, \quad \beta_{3,5} = \frac{b_1\beta_{2,2} + \beta_{2,4} + \beta_{2,6}}{\widehat{\mathcal{R}}(5)},$$
$$\beta_{3,6} = \frac{b_1 b_2}{\widehat{\mathcal{R}}(6)}, \quad \beta_{3,7} = \frac{b_1\beta_{2,4} + \beta_{2,6}}{\widehat{\mathcal{R}}(7)}, \quad \beta_{3,9} = \frac{b_1\beta_{2,6}}{\widehat{\mathcal{R}}(9)}.$$

Proceeding to $\mathcal{O}(\epsilon^4)$ and beyond, we use (19) to derive the general formula for $\beta_{n,j}$

$$\begin{aligned}\beta_{n,j} = \frac{1}{\widehat{\mathcal{R}}(j)}\Big(&(\delta_{j,2}+\delta_{j,4})b_{n-1}+\delta_{j,6}b_1b_{n-1}+(1-\delta_{j,2}-\delta_{j,4})\beta_{n-1,j-1}\\ &+(1-\delta_{j,2})\beta_{n-1,j+1}+b_1\big((1-\delta_{j,2}-\delta_{j,4}-\delta_{j,6})\beta_{n-1,j-3}+\beta_{n-1,j+3}\big)\\ &+\sum_{l=2}^{n-2}\Big(s_{n-l}\beta_{l,j}+b_{n-l}\big((1-\delta_{|j-3|,1}-\delta_{j,6})\beta_{l,|j-3|}+\beta_{l,j+3}\big)\\ &+\sum_{\substack{k=2\\k\neq 3}}^{\infty}\beta_{n-l,k}\big((1-\delta_{|k-j|,1}-\delta_{k,j}-\delta_{|k-j|,3})\beta_{l,|k-j|}+\beta_{l,k+j}\big)+\delta_{j,6}b_{n-l}b_l\Big)\Big).\end{aligned} \tag{25}$$

Note that if j is 2, 4, or 6, then $\beta_{n,j}$ will depend on b_{n-1}, which will be unknown when solving for $\beta_{n,j}$, as b_{n-2} is the unknown at n-th order. This can be seen in the b_2 dependence of formulas for $\beta_{3,2}$, $\beta_{3,4}$, and $\beta_{3,6}$ above. Thus, we make the following definitions

$$\beta_{n,2} = \frac{b_{n-1}}{\widehat{\mathcal{R}}(2)} + \overline{\beta}_{n,2}, \quad \beta_{n,4} = \frac{b_{n-1}}{\widehat{\mathcal{R}}(4)} + \overline{\beta}_{n,4}, \quad \beta_{n,6} = \frac{b_1 b_{n-1}}{\widehat{\mathcal{R}}(6)} + \overline{\beta}_{n,6}.$$

Each $\overline{\beta}$ contains all the terms from its respective coefficient except the ones which depend on b_{n-1}.

The solvability condition (17) becomes

$$\begin{aligned}s_{n-1}+b_1(\beta_{n-1,2}+\beta_{n-1,4})+\beta_{n-1,2}+\sum_{l=2}^{n-2}\Big(b_l\big(\beta_{n-l,2}+\beta_{n-l,4}\big)\\ +\sum_{k=4}^{\infty}\frac{1}{2}\beta_{n-l,k}\big((1-\delta_{k,4})\beta_{l,k-1}+\beta_{l,k+1}\big)\Big)=0,\end{aligned}$$

where the sum over k starts from 4 instead of 2 because $\beta_{n-l,2}$ would be paired with $\beta_{l,1}$ and $\beta_{l,3}$, neither of which exist. However, $\beta_{n-1,2}$ and $\beta_{n-1,4}$ depend on b_{n-2}, and so the $\overline{\beta}$ substitutions are made. Additionally, $b_{n-2}(\beta_{2,2}+\beta_{2,4})$ are extracted from the sum. This leaves

$$\begin{aligned}s_{n-1}+b_{n-2}\Big(2b_1\Big(\frac{1}{\widehat{\mathcal{R}}(2)}+\frac{1}{\widehat{\mathcal{R}}(4)}\Big)+\frac{3}{2\widehat{\mathcal{R}}(2)}\Big)\\ =-\Big(b_1\big(\overline{\beta}_{n-1,2}+\overline{\beta}_{n-1,4}\big)+\overline{\beta}_{n-1,2}+\sum_{l=2}^{n-3}\Big(b_l\big(\beta_{n-l,2}+\beta_{n-l,4}\big)\Big)\\ +\sum_{l=2}^{n-2}\Big(\sum_{k=4}^{\infty}\frac{1}{2}\beta_{n-l,k}\big((1-\delta_{k,4})\beta_{l,k-1}+\beta_{l,k+1}\big)\Big)\Big).\end{aligned} \tag{26}$$

Proceeding to the solvability condition (18), it becomes

$$b_1 s_{n-1} + \beta_{n-1,2} + \beta_{n-1,4} + b_1\beta_{n-1,6} + \sum_{l=2}^{n-2}\Big(s_{n-l}b_l + b_l\beta_{n-l,6} + \sum_{\substack{k=2\\k\neq 3}}^{\infty}\frac{1}{2}\beta_{n-l,k}\big((1-\delta_{|k-3|,1}-\delta_{k,6})\beta_{l,k-3}+\beta_{l,k+3}\big)\Big) = 0.$$

As was done with (26), $\overline{\beta}$ substitutions are made, and the terms $b_{n-2}s_2$ and $b_{n-2}\beta_{2,6}$ are separated from the sum. This gives a final form of (18) as

$$\begin{aligned}\Big(s_2 + \frac{3b_1^2}{2\widehat{\mathcal{R}}(6)} + \frac{1}{\widehat{\mathcal{R}}(2)} + \frac{1}{\widehat{\mathcal{R}}(4)}\Big)b_{n-2} + b_1 s_{n-1} \\ = -\Big(\overline{\beta}_{n-1,2} + \overline{\beta}_{n-1,4} + b_1\overline{\beta}_{n-1,6} + \sum_{l=2}^{n-3}\Big(b_l\big(s_{n-l}+\beta_{n-l,6}\big)\Big) \\ + \sum_{l=2}^{n-2}\Big(\sum_{\substack{k=2\\k\neq 3}}^{\infty}\frac{1}{2}\beta_{n-l,k}\big((1-\delta_{|k-3|,1}-\delta_{k,6})\beta_{l,|k-3|}+\beta_{l,k+3}\big)\Big)\Big).\end{aligned} \tag{27}$$

Expressions (26)–(27) give the solvability conditions from which s_{n-1} and b_{n-2} are determined. These values, together with (25), give all the unknown coefficients to construct a solution for $N = 3$. The solvability conditions are linear in the unknown parameters in b_{n-2} and s_{n-1}, and of the form

$$\begin{pmatrix} s_2 + \frac{3b_1^2}{2\widehat{\mathcal{R}}(6)} + \frac{1}{\widehat{\mathcal{R}}(2)} + \frac{1}{\widehat{\mathcal{R}}(4)} & b_1 \\ 1 & \Big(2b_1\Big(\frac{1}{\widehat{\mathcal{R}}(2)} + \frac{1}{\widehat{\mathcal{R}}(4)}\Big) + \frac{3}{2\widehat{\mathcal{R}}(2)}\Big)\end{pmatrix}\begin{pmatrix} b_{n-2} \\ s_{n-1}\end{pmatrix} = \vec{G}_n. \tag{28}$$

For the model equations examined in this work shown in Table 1, the above matrix is invertible, thus we have formal existence of the perturbation series to all orders.

4 Solution for Resonance at $N \geq 4$

In this section the solution for $N \geq 4$ is presented. The solution takes the same form for all $N \geq 4$; each value of N need not be treated separately in this range. While the method is the same as for $N = 3$, there are some key differences which must be addressed.

Naturally, we must change all instances of ϕ_3 to ϕ_N, as the condition for resonance is now $\widehat{\mathcal{R}}(1) = \widehat{\mathcal{R}}(N) = 0$ instead of $\widehat{\mathcal{R}}(1) = \widehat{\mathcal{R}}(3) = 0$. This transforms f_1, B, and F (Eqs. (6), (9), and (10), respectively) like so

$$f_1 = \cos(x) + \cos(Nx), \tag{29}$$

$$B(x;\epsilon) = \sum_{n=2}^{\infty}\epsilon^n B_n = \sum_{n=2}^{\infty}\epsilon^n b_n\phi_N, \tag{30}$$

$$F(x;\epsilon) = \sum_{n=2}^{\infty} \epsilon^n F_n = \sum_{n=2}^{\infty} \epsilon^n \sum_{\substack{k=2 \\ k \neq N}} \beta_{n,k} \phi_k, \tag{31}$$

and the solvability conditions (17) and (18) to

$$\langle \mathcal{R}[F_n], \phi_1 \rangle = 0, \tag{32}$$
$$\langle \mathcal{R}[F_n], \phi_N \rangle = 0. \tag{33}$$

The form of s given in (11) stays the same, as well as the forms of $\mathcal{R}[F_2]$ and $\mathcal{R}[F_n]$ given in (15) and (16).

Starting with $\mathcal{O}(\epsilon^2)$, and using (15), we find

$$\mathcal{R}[F_2] = s_1\phi_1 + \frac{1}{2}\phi_2 + b_1\phi_{N-1} + s_1 b_1 \phi_N + b_1 \phi_{N+1} + \frac{1}{2} b_1^2 \phi_{2N}. \tag{34}$$

The above equation is what leads to the distinction between $N = 3$ and $N \geq 4$. For $N = 3$, the ϕ_2 term shares a wavenumber with the ϕ_{N-1} term, producing (20). This changes the forms of the solutions compared to $N \geq 4$.

Substituting (34) into the solvability conditions (32) and (33) gives

$$\langle \mathcal{R}[F_2], \phi_1 \rangle = \pi s_1 = 0,$$
$$\langle \mathcal{R}[F_2], \phi_N \rangle = \pi s_1 b_1 = 0.$$

The solution is $s_1 = 0$, with b_1 arbitrary, just like with $N = 3$. Substituting this back into (34) gives

$$\mathcal{R}[F_2] = \frac{1}{2}\phi_2 + b_1\phi_{N-1} + b_1\phi_{N+1} + \frac{1}{2} b_1^2 \phi_{2N}.$$

The coefficients of the solution F_2 can be written explicitly as,

$$\beta_{2,2} = \frac{1}{2\widehat{\mathcal{R}}(2)}, \quad \beta_{2,N-1} = \frac{b_1}{\widehat{\mathcal{R}}(N-1)},$$
$$\beta_{2,N+1} = \frac{b_1}{\widehat{\mathcal{R}}(N+1)}, \quad \beta_{2,2N} = \frac{b_1^2}{2\widehat{\mathcal{R}}(2N)}.$$

These results are identical to those produced for $N = 3$, excepting that $\beta_{2,2}$ and $\beta_{2,N-1}$ no longer correspond to the same wavenumber.

Proceeding to the next order ($O(\epsilon^3)$), we use (16) and the form of F_2 found above to write the solvability conditions (32) and (33) as

$$\langle \mathcal{R}[F_3], \phi_1 \rangle = \pi \left(s_2 + \beta_{2,2} + b_1\beta_{2,N-1} + b_1\beta_{2,N+1} \right) = 0,$$
$$\langle \mathcal{R}[F_3], \phi_N \rangle = \pi \left(s_2 b_1 + \beta_{2,N-1} + \beta_{2,N+1} + b_1\beta_{2,2N} \right) = 0.$$

Substituting in the formulas for each β gives the system of equations

$$s_2 + \frac{1}{2\widehat{\mathcal{R}}(2)} + \left(\frac{1}{\widehat{\mathcal{R}}(N-1)} + \frac{1}{\widehat{\mathcal{R}}(N+1)}\right) b_1^2 = 0, \tag{35}$$

$$\left(s_2 + \frac{1}{\widehat{\mathcal{R}}(N-1)} + \frac{1}{\widehat{\mathcal{R}}(N+1)}\right) b_1 + \left(\frac{1}{2\widehat{\mathcal{R}}(2N)}\right) b_1^3 = 0. \tag{36}$$

As the above is cubic in b_1, it has three possible solutions, as with $N = 3$. Regardless of the model, $b_1 = 0$ and $s_2 = -1/2\widehat{\mathcal{R}}(2)$ is a solution. For the operators in Table 1, the other solutions are imaginary and are not considered in this work. The conditions required to produce non-zero real values of b_1 are discussed in Sect. 6. Using $b_1 = 0$ and $s_2 = -1/2\widehat{\mathcal{R}}(2)$, the remaining part of $\mathcal{R}[F_3]$ is

$$\mathcal{R}[F_3] = \beta_{2,2}\phi_3 + b_2\phi_{N-1} + b_2\phi_{N+1}.$$

Therefore, the non-resonant coefficients at $\mathcal{O}(\epsilon^3)$ are

$$\beta_{3,3} = \frac{\beta_{2,2}}{\widehat{\mathcal{R}}(3)} = \frac{1}{2\widehat{\mathcal{R}}(2)\widehat{\mathcal{R}}(3)}, \quad \beta_{3,N-1} = \frac{b_2}{\widehat{\mathcal{R}}(N-1)}, \quad \beta_{3,N+1} = \frac{b_2}{\widehat{\mathcal{R}}(N+1)}.$$

Note that, for $N = 4$, $\beta_{3,3} = \beta_{3,N-1} = \beta_{2,2}/R(3) + b_2/R(N-1)$, which leads to b_2 being the first non-zero resonant mode coefficient for $N = 4$.

The unknowns at $\mathcal{O}(\epsilon^n)$ are s_{n-1}, b_{n-2}, and the function F_n, and we now proceed to calculate their general form. Using $\mathcal{R}[F_n]$ given by (16) with the solvability conditions for s_{n-1} and b_{n-2} and (19) for each $\beta_{n,j}$ yields

$$\begin{aligned}\beta_{n,j} &= \frac{1}{\pi\widehat{\mathcal{R}}(j)}\Big(\langle s_{n-1}f_1 + (2f_1 + s_1)(B_{n-1} + F_{n-1}) + \sum_{l=2}^{n-2}(s_{n-l} + F_{n-l} + B_{n-l})(F_l + B_l), \phi_j\rangle\Big) \\ &= \frac{1}{\pi\widehat{\mathcal{R}}(j)}\Big(\langle s_{n-1}f_1, \phi_j\rangle + \langle 2f_1B_{n-1}, \phi_j\rangle + \langle 2f_1F_{n-1}, \phi_j\rangle + \langle s_1B_{n-1}, \phi_j\rangle + \langle s_1F_{n-1}, \phi_j\rangle + \\ &\quad \sum_{l=2}^{n-2}\big(\langle s_{n-l}F_l, \phi_j\rangle + \langle s_{n-l}B_l, \phi_j\rangle + \langle F_{n-l}F_l, \phi_j\rangle + 2\langle F_{n-l}B_l, \phi_j\rangle + \langle B_{n-l}B_l, \phi_j\rangle\big)\Big).\end{aligned} \tag{37}$$

Using the expressions in (11), (29), (30) and (31), which give the representation of the unknowns as a series in ϵ, we find the general form of $\beta_{n,j}$ as

$$\begin{aligned}\beta_{n,j} = \frac{1}{\widehat{\mathcal{R}}(j)}\Bigg(&(\delta_{j,N-1} + \delta_{j,N+1})b_{n-1} + (1 - \delta_{j,2} - \delta_{j,N+1})\beta_{n-1,j-1} + (1 - \delta_{j,N-1})\beta_{n-1,j+1} \\ &+ \sum_{l=2}^{n-2}\Bigg(s_{n-l}\beta_{l,j} + \sum_{\substack{k=2,\\k\neq N}}^{\infty}\frac{1}{2}\beta_{n-l,k}\Big((1 - \delta_{k,j} - \delta_{|k-j|,1} - \delta_{|k-j|,N})\beta_{l,|k-j|} + (1 - \delta_{k+j,N})\beta_{l,k+j}\Big)\end{aligned}$$

$$+ b_l\left((1-\delta_{|j-N|,1}-\delta_{j,2N})\beta_{n-l,|j-N|}+\beta_{n-l,j+N}\right)+\delta_{j,2N}\frac{1}{2}b_{n-l}b_l\Bigg)\Bigg)\Bigg). \tag{38}$$

Note that (38) for $j = N \pm 1$ contains b_{n-1}, which we do not yet have at this order. This implies that $\beta_{n-1,N-1}$ and $\beta_{n-1,N+1}$, the coefficients neighboring the resonant mode at previous order, will depend on b_{n-2}, the unknown at $\mathcal{O}(\epsilon^n)$. Thus, we define

$$\begin{aligned}\beta_{n,N-1} &= \frac{1}{\widehat{\mathcal{R}}(N-1)}\Bigg(b_{n-1}+\beta_{n-1,N-2}+\sum_{l=2}^{n-2}\Bigg(s_{n-l}\beta_{l,N-1}+b_l\beta_{n-l,2N-1}\\ &\quad+\sum_{\substack{k=2,\\k\neq N}}^{\infty}\frac{1}{2}\beta_{n-l,k}\left((1-\delta_{k,N-1}-\delta_{k,N-2}-\delta_{k,2N-1})\beta_{l,|k-N+1|}+\beta_{l,k+N-1}\right)\Bigg)\Bigg)\\ &= \frac{b_{n-1}}{\widehat{\mathcal{R}}(N-1)}+\overline{\beta}_{n,N-1}\end{aligned} \tag{39}$$

and

$$\begin{aligned}\beta_{n,N+1} &= \frac{1}{\widehat{\mathcal{R}}(N+1)}\Bigg(b_{n-1}+\beta_{n-1,N+2}+\sum_{l=2}^{n-2}\Bigg(s_{n-l}\beta_{l,N+1}+b_l\beta_{n-l,2N+1}\\ &\quad+\sum_{\substack{k=2,\\k\neq N}}^{\infty}\frac{1}{2}\beta_{n-l,k}\left((1-\delta_{k,N+1}-\delta_{k,N+2}-\delta_{k,2N+1})\beta_{l,|k-N-1|}+\beta_{l,k+N+1}\right)\Bigg)\Bigg)\\ &= \frac{b_{n-1}}{\widehat{\mathcal{R}}(N+1)}+\overline{\beta}_{n,N+1},\end{aligned} \tag{40}$$

where we introduced $\overline{\beta}_{n-1,N\pm1}$, analogous to the $\overline{\beta}$ introduced for the solution with $N = 3$.

Proceeding to the first solvability condition (32), and again substituting (11), (29), (30), (31), and (16), we obtain an equation which may be rearranged to give the corrections to the wave speed

$$\begin{aligned}s_{n-1} &= -\Bigg(\beta_{n-1,2}+\sum_{l=2}^{n-2}\Bigg(b_l\left(\beta_{n-l,N-1}+\beta_{n-l,N+1}\right)\\ &\quad+\sum_{\substack{k=2,\\k\neq N}}^{\infty}\frac{1}{2}\beta_{n-l,k}\left((1-\delta_{k,2}-\delta_{k,N+1})\beta_{l,k-1}+(1-\delta_{k,N-1})\beta_{l,k+1}\right)\Bigg)\Bigg),\end{aligned} \tag{41}$$

and doing so for the second solvability condition (33) gives the coefficients of the resonant mode

$$b_{n-2} = -\frac{1}{s_2}\left(\beta_{n-1,N-1} + \beta_{n-1,N+1} + \sum_{l=2}^{n-3}\Big(b_l\big(s_{n-l} + \beta_{n-l,2N}\big)\Big)\right.$$
$$\left. + \sum_{l=2}^{n-2}\left(\sum_{\substack{k=2\\k\neq N}}^{\infty}\frac{1}{2}\beta_{n-l,k}\left((1-\delta_{|N-k|,1})\beta_{l,|N-k|} + \beta_{l,N+k}\right)\right)\right). \tag{42}$$

As (42) contains $\beta_{n-1,N-1}$ and $\beta_{n-1,N+1}$, which depend on b_{n-2}, the $\overline{\beta}$ substitutions given by (40) and (40) must be made. Using these as well as $s_2 = -\frac{1}{2\widehat{\mathcal{R}}(2)}$, b_{n-2} can be isolated as

$$\left(\frac{1}{2\widehat{\mathcal{R}}(2)} - \frac{1}{\widehat{\mathcal{R}}(N-1)} - \frac{1}{\widehat{\mathcal{R}}(N+1)}\right)b_{n-2} = -\left(\overline{\beta}_{n-1,N-1} + \overline{\beta}_{n-1,N+1}\right.$$
$$\left. + \sum_{l=2}^{n-3}\Big(b_l\big(s_{n-l} + \beta_{n-l,2N}\big)\Big) + \sum_{l=2}^{n-2}\left(\sum_{\substack{k=2\\k\neq N}}^{\infty}\frac{1}{2}\beta_{n-l,k}\Big((1-\delta_{|N-k|,1})\beta_{l,|N-k|} + \beta_{l,N+k}\Big)\right)\right). \tag{43}$$

Equations (38), (41) and (43) represent recursive relations which can be used to calculate the correction to the wave speed and the Fourier coefficients at $\mathcal{O}(\epsilon^n)$. These expressions can be further simplified by invoking the following theorem:

Theorem 1 *Assuming the series expansion seen in (29)–(31) for an asymptotic solution to Eq. (3) with a quadratic nonlinearity, for $k, m \in \mathbb{Z}$, we have the following dichotomy*

- *at odd order in the small parameter, $O(\epsilon^{2k+1})$, the functions F_{2k+1}, B_{2k+1} contain only modes with odd wavenumber, ϕ_{2m+1},*
- *at even order in the small parameter, $O(\epsilon^{2k})$, the functions F_{2k}, B_{2k} contain only modes with even wavenumber, ϕ_{2m}.*

Below we outline the proof, which can be done by strong induction directly from formulas (38), (41) and (42) with the details in Appendix B. This pattern arises due to quadratic nonlinearity giving the cosine identity (12). We know the modes at $\mathcal{O}(\epsilon^n)$ are produced by multiplication of the solution at $\mathcal{O}(\epsilon^m)$ and $\mathcal{O}(\epsilon^{n-m})$ $\forall\ m < n$, and they are the sums and differences of the modes at those orders. Alongside the initial condition $f_1 = \phi_1$ (or $f_1 = \phi_1 + b_1\phi_3$ in the case of $N = 3$), which obeys Theorem 1, this can be used as a base case for an induction proof using the following reasoning:

- If n is even, then m and $n - m$ have the same parity. Assuming that both $\mathcal{O}(\epsilon^m)$ and $\mathcal{O}(\epsilon^{n-m})$ obey Theorem 1, the modes present at these orders will also have the same parity. Thus the sums and differences of the modes will be even, and only even modes will be found at $\mathcal{O}(\epsilon^n)$.

- If n is odd, then m and $n-m$ have different parities. Assuming that both $\mathcal{O}(\epsilon^m)$ and $\mathcal{O}(\epsilon^{n-m})$ obey Theorem 1, the modes present at these orders will also have different parities. Thus, the sums and differences of the modes will be odd, and only odd modes will be found at $\mathcal{O}(\epsilon^n)$.

Through similar logic, we may also make a statement about the maximal wavenumber in the support of the corrections in (10):

Theorem 2 *For the function F_n, the $\mathcal{O}(\epsilon^n)$ correction to a ripple with resonant mode $N \geq 3$, the largest k for which ϕ_k has a non-zero coefficient is*

$$K_N(n) = n + 2\,\text{floor}\left(\frac{n}{N-2}\right) = n + 2\left\lfloor \frac{n}{N-2} \right\rfloor. \tag{44}$$

Figure 3 gives a graphical representation of Theorem 2. Similar to the previous proof, this one can also be done by strong induction. While the details are included in Appendix B, we proceed to outline the proof here. Typically, as order n is increased, the largest non-zero mode k included in the Stokes expansion represented by (31) also increases by 1. However, at orders which are multiples of $N-2$, the highest non-zero mode is increased by 3, where we refer to the extra 2 modes as a **boost**. The first boost arises because at $\mathcal{O}(\epsilon^N)$ we solve for a non-zero coefficient of ϕ_N for the first time. However, the unknown coefficient of ϕ_N at this order is b_{N-2}, and thus at $\mathcal{O}(\epsilon^{N-2})$ the highest mode is ϕ_N. All subsequent boosts then occur at multiples of $N-2$.

We note that the correction to the wave speed s_{n-1} can be thought of as the coefficient of the zeroth mode as $\phi_0 = 1$. As such, it will also follow Theorem 1, meaning that s_{n-1} will only be non-zero if n is odd, i.e., when $n-1$ is even, leading to the following lemma

Lemma 3 *The corrections to the wave speed c contain only even powers of the small parameter ϵ, i.e., $s_{2k+1} = 0$ and therefore*

$$c = c_0 + \sum_{n=1}^{\infty} \epsilon^{2n} s_{2n}. \tag{45}$$

Similarly, b_{n-2} is solved for using the second solvability condition (33), which considers the N-th mode, and so may only be non-zero when n and N have the same parity, which is also when $n-2$ and N have the same parity. Applying Theorem 1 results in:

Lemma 4 *For resonance occurring at an odd integer N, the resonant mode ϕ_N will only be present at odd orders of the small parameter ϵ (i.e., $b_{2k} = 0$). For resonance occurring at an even integer N, the resonant mode ϕ_N will only be present at even powers of the small parameter ϵ (i.e., $b_{2k+1} = 0$).*

Whereas using Theorem 2 to understand when ϕ_N first appears (i.e., by solving $K_N(n) = N$ for n), results in:

Lemma 5 *When the Wilton ripple is at mode N, the first non-zero coefficient for ϕ_N is found at $\mathcal{O}(\epsilon^{N-2})$, and is given by b_{N-2}.*

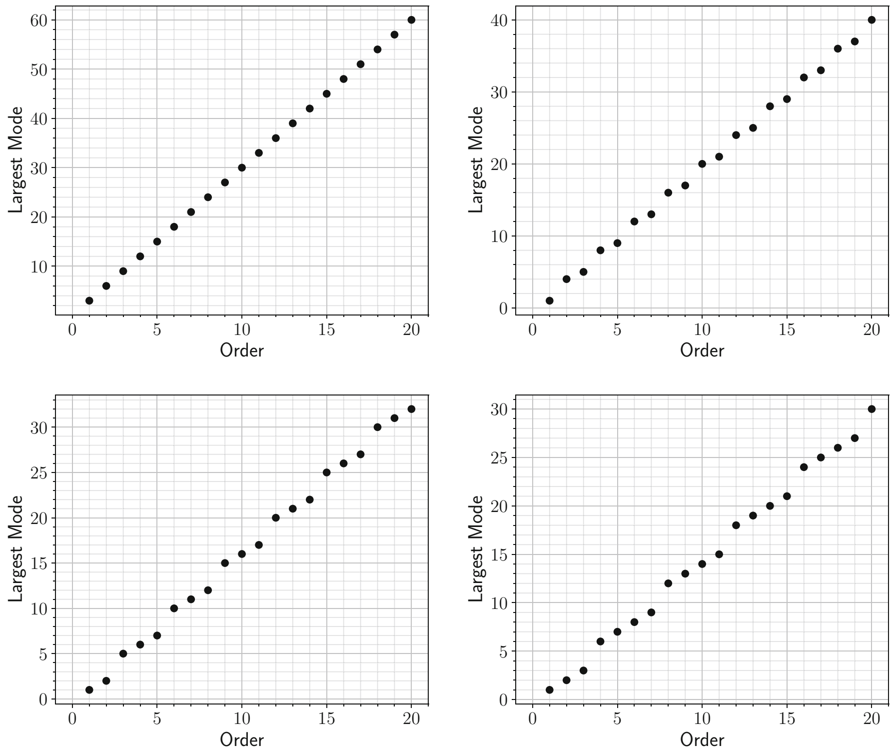
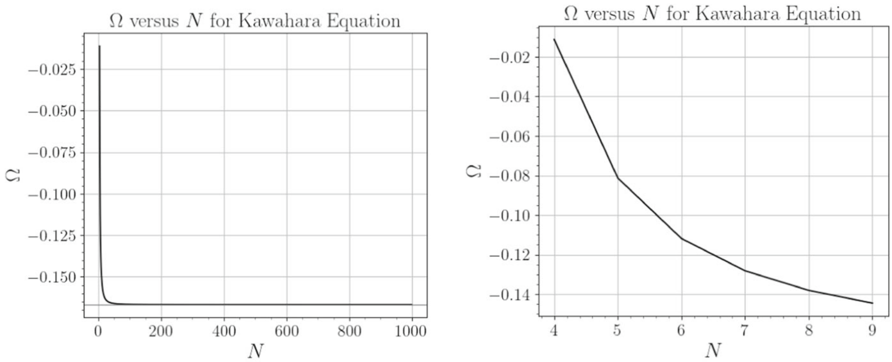


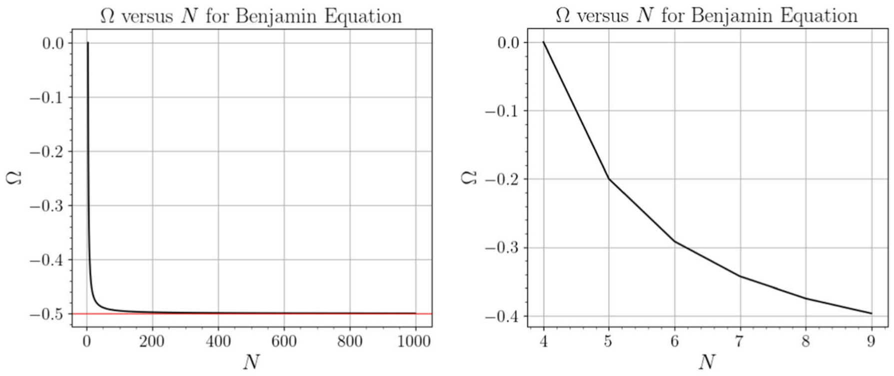
Fig. 3 Graphical representation of the dependence on largest non-zero mode m on the order of the small parameter n obtained from the general formulas (38) for resonances at $N = 3, 4, 5, 6$ confirming Theorem 2. These plots show a linear dependence unless n is divisible by $N - 2$, at which point there is a boost of 2 in the number of modes needed to capture the series solution



(a) Kawahara equation, $N = 4$ to 1000.

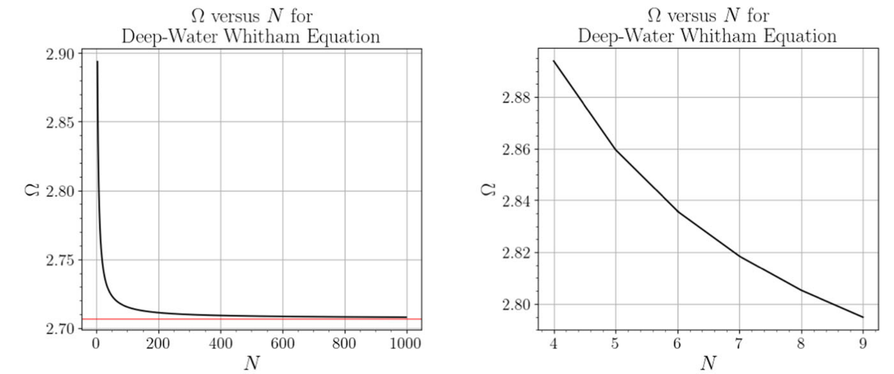
(b) Kawahara equation, $N = 4$ to 10.

Fig. 4 The zoomed out (**a**) and zoomed in (**b**) plots of the common divisor $\Omega(N)$ seen in (46) for the Kawahara equation. For $N = 4$, it is close to zero, but in the limit $N \to \infty$, it approaches $-1/6$



(a) Benjamin equation, $N = 4$ to 1000. (b) Benjamin equation, $N = 4$ to 10.

Fig. 5 The zoomed out (**a**) and zoomed in (**b**) plots of the common divisor $\Omega(N)$ seen in (46) for the Benjamin equation. For $N = 4$, it is exactly zero, but in the limit $N \to \infty$, it approaches $-1/2$



(a) Deep-water Whitham equation, $N = 4$ to 1000. (b) Deep-water Whitham equation, $N = 4$ to 10.

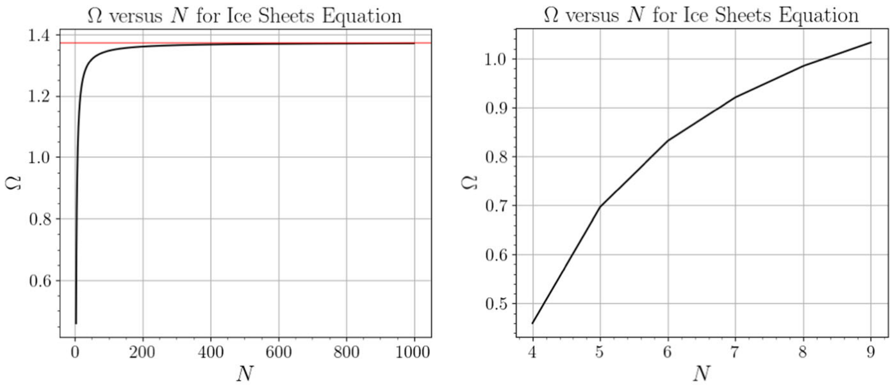
Fig. 6 The zoomed out (**a**) and zoomed in (**b**) plots of the common divisor $\Omega(N)$ seen in (46) for the deep-water Whitham equation. In the limit $N \to \infty$, $\Omega(N) \approx 2.707$

5 Common Divisor

An important feature identified here is the recurring divisor in (43). It was shown that at every order, the unknown b_{n-2} has a coefficient of

$$\Omega(N) = \frac{1}{2\widehat{\mathcal{R}}(2)} - \frac{1}{\widehat{\mathcal{R}}(N-1)} - \frac{1}{\widehat{\mathcal{R}}(N+1)}. \tag{46}$$

This coefficient is fixed once N and $\mathcal{L}$ have been chosen. For some choices of $\mathcal{L}$, the recurring divisor $\Omega(N)$ in (46) can get very small. Different $\Omega(N)$ are depicted in Figs. 4, 5, 6 and 7 for each value of N from 4 to 1000 in (a) and 4 to 10 in (b) for each $\mathcal{L}$ given by equations see in Table 1. One special case is the value of Ω for the Benjamin equation seen in Fig. 5 at $N = 4$, which is exactly 0, leading to a failure of the solution method; when $\Omega = 0$ the series does not exist even formally. Another interesting case is the Kawahara



(a) Hydroelastic equation, $N = 4$ to 1000. (b) Hydroelastic equation, $N = 4$ to 10.

Fig. 7 The zoomed out (**a**) and zoomed in (**b**) plots of the common divisor $\Omega(N)$ seen in (46) for the hydroelastic equation. In the limit $N \to \infty$, $\Omega(N) \approx 1.374$

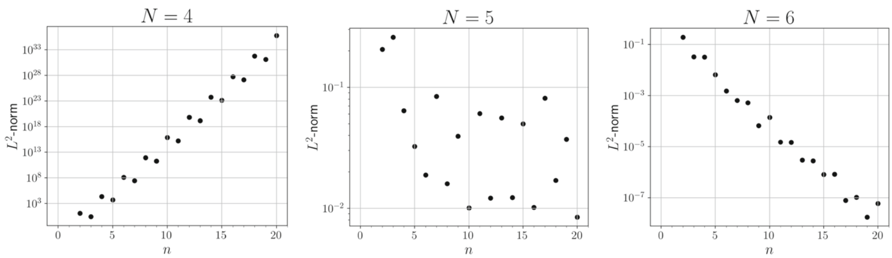


Fig. 8 The L^2-norm of the $O(\epsilon^n)$ correction, $\|F_n(x) + B_N(x)\|_2$, for the Kawahara equation with $N = 4$ (left), $N = 5$ (center), and $N = 6$ (right)

equation with $N = 4$, shown in Fig. 4, where Ω gets close to 0, which reduces the radius of convergence for the asymptotic series (explored further in Sect. 5.1).

For the other linear operators in Table 1, Ω does not get particularly close to 0 for any N. As $N \to \infty$, Ω approaches $1/6$ for the Kawahara equation, $-1/2$ for the Benjamin equation, $(3 - \sqrt{2})/(2 - \sqrt{2}) \approx 2.707$ for the deep-water Whitham equation, and $(1 + \sqrt{2})/(6 - 3\sqrt{2}) \approx 1.374$ for the hydroelastic equation, as shown in Figs. 4, 5, 6 and 7, respectively. For the Akers–Milewski equation, Ω is 1 for all values of N.

5.1 Example: The Kawahara Equation $N \geq 4$

As discussed above, the recurring divisor (46) for the Kawahara equation with $N = 4$ is small, which means the method produces accurate solutions for a much smaller range of ϵ than the equations with different values of N or different choices of $\mathcal{L}$. Figure 8 depicts the L^2-norms of $F_n + B_n$ (i.e., the solution at $\mathcal{O}(\epsilon^n)$) versus n for the Kawahara equation with $N = 4$, 5 and 6 where n goes from 1 to 20. In order for the solution to converge, as n increases, ϵ^n must decrease faster than the coefficients of the solution at $\mathcal{O}(\epsilon^n)$ increase. Since Fig. 8 is semilog in y, the slope of the L^2-norm with respect to n must be less than $|\log_{10}(\epsilon)|$ to have convergence. The Kawahara solutions for $N = 5$ and $N = 6$ have slopes less than or equal to 0, so they are valid for all $\epsilon \leq 1$, though of course when using truncated solutions, the error will be greater for larger ϵ. However, for $N = 4$, the coefficients grow quickly, with the slope of the line of best fit being approximately 1.787. Thus, the solution is only valid when $\epsilon \leq 10^{-1.787} = 0.0163$, which is a relatively small radius of convergence. Figure 9 shows sample solutions for $N = 4, 5, 6$ with different $\epsilon = 0.01, 0.1, 0.5$. We see that when $N = 4$, for values of $\epsilon = 0.1, 0.5$ which are outside of the radius of convergence, the amplitude of the solution gets very large and the ripples take over the main cosine-like profile. Comparatively, the solutions for $N = 5$ and $N = 6$ converge for all values of ϵ, and in fact converge for even larger values of ϵ than those shown. This highlights the effect of a small recurring divisor in (46) on the radius of convergence.

In the work by Akers and Nicholls [17], similar models were examined for the case of resonance at $N = 2$ and two model features were evaluated as predictors of convergence rates. The first was the spectral gap, defined as the smallest difference between the value of the linear operator for the first mode and any other non-resonant mode, or rather the smallest value of $\widehat{\mathcal{R}}(k)$, i.e.,

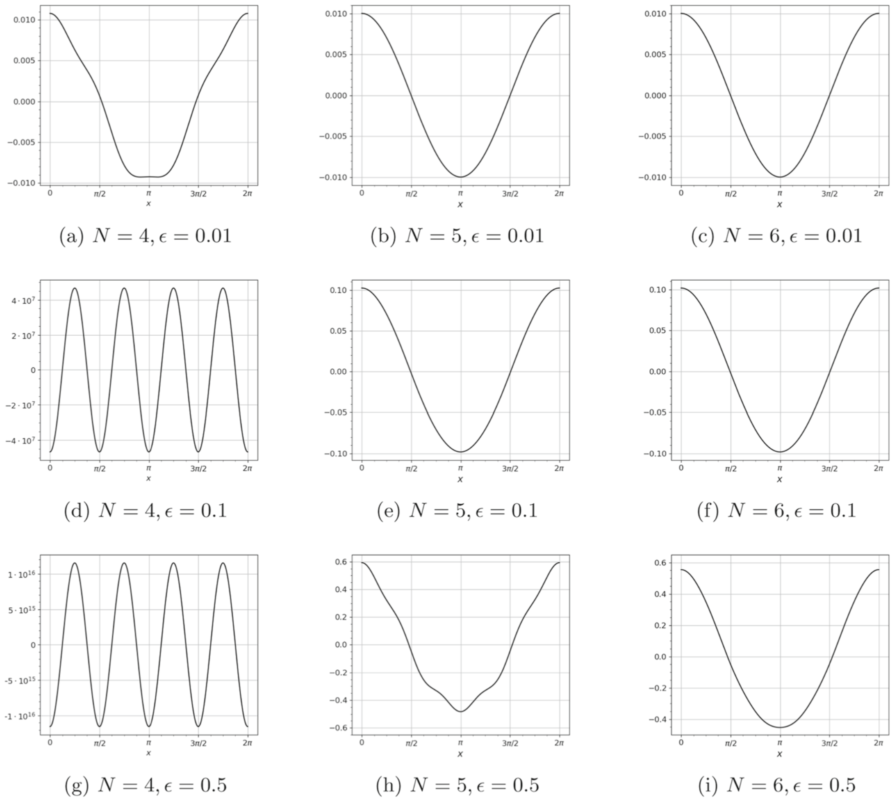
$$\min_{\substack{k\geq 2\\ k\neq N}} |\widehat{\mathcal{L}}(k) - \widehat{\mathcal{L}}(1)| = \min_{\substack{k\geq 2\\ k\neq N}} |\widehat{\mathcal{R}}(k)|.$$

$\widehat{\mathcal{R}}(k)$ is the divisor when solving for $\beta_{n,k}$, and so the spectral gap acts as a lower bound on the magnitude of that divisor. Thus, one might expect larger spectral gaps to contribute to convergence rates. The second feature was the limiting behavior of the linear operator at infinity; operators with faster growth at infinity are more smoothing when inverted. Neither was observed to be completely predictive.

For the Kawahara with $N = 4$, both the spectral gap (≈ 2.11) and the high frequency asymptotic ($\mathcal{O}(k^4)$), are fairly large [17] when compared to the same parameters for the other models. This suggests that they are not what is leading to the small radius of convergence that is seen in this work, and instead the small recurring divisor Ω seems to be a better indicator. We conjecture that the recurring divisor Ω should be considered alongside the spectral gap and the high frequency asymptotic when estimating the radius of convergence.

6 Conclusion and Future Work

In this work, we examined asymptotic solutions to nonlinear, dispersive equations given by (1) with the linear operators shown in Table 1. Considering periodic traveling wave solutions, we simplified the PDE and wrote it in the form of (3). The specific case of the Kawahara model with $N = 3$ was considered in Sect. 2 to present the solution method. The first three orders were solved using the representation shown in (8), with the first-order solution given by (6), higher-order resonances contained in (9), and the wavespeed correction given by (11). In Sect. 3, the findings of Sect. 2 were extended to any model, though still for $N = 3$, and this generalization was used to derive recursive formulas for the wavespeed correction and the coefficient of each mode at every order. In Sect. 4, we considered $N \geq 4$, and again derived formulas similar to the ones found in Sect. 3. We also presented two theorems relating to what modes are present at each order—one theorem stating that the parity of the order matches the parity of the modes present at that order, and



(a) $N = 4, \epsilon = 0.01$ (b) $N = 5, \epsilon = 0.01$ (c) $N = 6, \epsilon = 0.01$

(d) $N = 4, \epsilon = 0.1$ (e) $N = 5, \epsilon = 0.1$ (f) $N = 6, \epsilon = 0.1$

(g) $N = 4, \epsilon = 0.5$ (h) $N = 5, \epsilon = 0.5$ (i) $N = 6, \epsilon = 0.5$

Fig. 9 Periodic traveling waves with resonances at different modes with increasing amplitude of solutions to the Kawahara equation. Solutions are taken up to $\mathcal{O}(\epsilon^6)$. We see that due to the recurrent divisor in (46) becoming small for $N = 4$, the amplitude of the solutions grow very fast as ϵ increases, and the radius of convergence is small. For $N = 5, 6$, the solutions stay bounded. We note that (d) and (g) are outside of the radius of convergence which was calculated to be approximately 0.0163, but are included here for illustrative purposes

another providing a formula for the highest mode present at each order. These theorems allow us to greatly simplify the formulas found in Sect. 4, and the simplified formulas are presented in Appendix A. One of the most important results of this work is understanding that the first time resonance N appears with a non-zero coefficient is at order $N-2$, which is described in Lemma 5.

From previous work [1, 2], we know for $k = 2$, the amplitude of the resonant harmonics are on the same order of magnitude as the main wave and these were seen in experiments [13, 14]. The results presented in this paper show that for $k = N$ where $N > 3$, the amplitude of resonances are at order ϵ^{N-2} and for $k = 3$, the amplitude of the resonant solution is an order less than that of the main wave. That means these kinds of resonances will be much harder to find, especially given that in the present theoretical work we do not consider dissipation that would be present in a physical setting.

As we saw, for some linear operators examined in this work in Sect. 5, a recurrent coefficient needed to compute the resonant coefficient given by Ω in (46) can get quite small. While the algorithm used is effective for the majority of equations investigated in this paper, there are linear operators where $\Omega = 0$, in which case the method fails completely, such as the one for the Benjamin equation when $N = 4$. When Ω is merely small, such as with the Kawahara equation with $N = 4$, renormalization may be possible. Relating the methodology explored here with more commonly used Lindstedt–Poincaré method [32] could give a more formal way of exploring where certain small parameters in the problem approach zero and secular growths appear.

Another interesting conclusion is that while the operators $\mathcal{L}$ investigated in this paper did not produce any non-zero, real solutions for b_1 when $N \geq 4$, this is not necessarily prohibited for the general $\mathcal{L}$. Recall that b_1 is determined from the solution for the system of equations given by (35) and (36). Solving (35) for s_2 and substituting the result into (36) produces

$$b_1\left(b_1^2\left(\frac{1}{2\widehat{\mathcal{R}}(2N)}-\frac{1}{\widehat{\mathcal{R}}(N+1)}-\frac{1}{\widehat{\mathcal{R}}(N-1)}\right)+\frac{1}{\widehat{\mathcal{R}}(N+1)}+\frac{1}{\widehat{\mathcal{R}}(N-1)}-\frac{1}{2\widehat{\mathcal{R}}(2)}\right)=0. \tag{47}$$

Equating the portion inside the bracket to zero gives solutions of

$$b_1=\pm\sqrt{\left(\frac{1}{2\widehat{\mathcal{R}}(2N)}-\frac{1}{\widehat{\mathcal{R}}(N+1)}-\frac{1}{\widehat{\mathcal{R}}(N-1)}\right)\left(\frac{1}{\widehat{\mathcal{R}}(N+1)}+\frac{1}{\widehat{\mathcal{R}}(N-1)}-\frac{1}{2\widehat{\mathcal{R}}(2)}\right)}. \tag{48}$$

For the solution to be real, the two brackets must either be both positive or both negative. This is clearly the case whenever $\frac{1}{\widehat{\mathcal{R}}(N+1)}+\frac{1}{\widehat{\mathcal{R}}(N-1)}$ is between $\frac{1}{2\widehat{\mathcal{R}}(2N)}$ and $\frac{1}{2\widehat{\mathcal{R}}(2)}$. Finding and exploring equations in which this condition is satisfied would be an intriguing next step.

We note that all the results presented in this work are valid for the quadratic nonlinearity where we used the cosine identities (12). It would also be interesting to consider equations like (1) modified by changing only the nonlinearity, that is, equations given by

$$u_t + \mathcal{L}u_x + (u^p)_x = 0, \tag{49}$$

with p an integer greater than 2 to see if a recurring divisor appears or if there are analogs to Theorems 1 and 2. In exploring model equations with higher nonlinearities, we can start to understand what would happen for full water wave equations where the nonlinearity is more complicated than just $p = 2$.

Finally, it is worth mentioning that in the very high-order limit, the Wilton ripples examined in this work start to resemble what is referred to as “generalized solitary waves” and have been examined in detail for both gravity-capillary waves and flexural-gravity waves, for example in [21, 33, 34]. However, no such formal analysis of their equivalence exists but it would be interesting to explore their relationship.

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Data Availability The data generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of Interest The authors have no relevant financial or non-financial interests to disclose.

Appendix A: Simplified Formulas

This section presents the formulas for s_{n-1}, b_{n-2}, and $\beta_{n,j}$ when $N \geq 4$ once Theorems 1 and 2 have been applied. The original formulas are given by (41), (43), and (38), respectively. The new formulas will depend on the parity of n and N (and j in the case of $\beta_{n,j}$), and thus separate formulas must be given for each parity. Fortunately, for many combinations of parities, the coefficients are always 0.

We begin with the formula for s_{n-1}. If n is even (and thus $n-1$ is odd), then s_{n-1} will be zero. If n is odd, then the formula depends on the parity of N. For even N, we find

$$s_{n-1} = -\Big(\beta_{n-1,2} + \sum_{l=1}^{\frac{n-3}{2}}\Big(b_{2l}(\beta_{n-2l,N-1} + \beta_{n-2l,N+1}) + \frac{1}{2}\sum_{\substack{k=1\\k\neq N/2}}^{K_N(2l)/2}\big(\beta_{2l,2k}((1-\delta_{k,1})\beta_{n-2l,2k-1} + \beta_{n-2l,2k+1})\big) + \frac{1}{2}\sum_{k=1}^{(K_N(2l+1)-1)/2}\big(\beta_{2l+1,2k+1}((1-\delta_{2k,N})\beta_{n-2l-1,2k} + (1-\delta_{2k+2,N})\beta_{n-2l-1,2k+2})\big)\Big)\Big) \tag{50}$$

and for odd N, we find

$$s_{n-1} = -\Big(\beta_{n-1,2} + \sum_{l=1}^{\frac{n-3}{2}}\Big(b_{2l+1}(\beta_{n-2l-1,N-1} + \beta_{n-2l-1,N+1}) + \frac{1}{2}\sum_{k=1}^{K_N(2l)/2}\big(\beta_{2l,2k}((1-\delta_{2k,N+1})\beta_{n-2l,2k-1} + (1-\delta_{2k,N-1})\beta_{n-2l,2k+1})\big) + \frac{1}{2}\sum_{\substack{k=1\\k\neq(N-1)/2}}^{(K_N(2l+1)-1)/2}\big(\beta_{2l+1,2k+1}(\beta_{n-2l-1,2k} + \beta_{n-2l-1,2k+2})\big)\Big)\Big). \tag{51}$$

For b_{n-2}, the result is zero if N and n do not have the same parity. For even N and n, the formula reduces to

$$\Omega(N)b_{n-2} = \overline{\beta}_{n-1,N-1} + \overline{\beta}_{n-1,N+1} + \sum_{l=1}^{\frac{n-2}{2}}\Big((1-\delta_{2l,n-2})b_{2l}(s_{n-2l} + \beta_{n-2l,2N})$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{\substack{k=1\\k\neq N/2}}^{K_N(2l)/2}\big(\beta_{2l,2k}(\beta_{n-2l,|N-2k|}+\beta_{n-2l,N+2k})\big)\\
&+(1-\delta_{2l,n-2})\frac{1}{2}\sum_{k=1}^{(K_N(2l+1)-1)/2}\\
&\big(\beta_{2l+1,2k+1}((1-\delta_{|N-2k-1|,1})\beta_{n-2l-1,|N-2k-1|}+\beta_{n-2l-1,N+2k+1})\big)\Big)
\end{aligned}
\tag{52}
$$

and for odd N and n, the formula reduces to

$$
\begin{aligned}
\Omega(N)b_{n-2}=\overline{\beta}_{n-1,N-1}+\overline{\beta}_{n-1,N+1}+&\sum_{l=1}^{\frac{n-3}{2}}\Big((1-\delta_{2l+1,n-2})b_{2l+1}(s_{n-2l-1}+\beta_{n-2l-1,2N})\\
&+\frac{1}{2}\sum_{k=1}^{K_N(2l)/2}\big(\beta_{2l,2k}((1-\delta_{|N-2k|,1})\beta_{n-2l,|N-2k|}+\beta_{n-2l,N+2k})\big)\\
&+\frac{1}{2}\sum_{\substack{k=1\\k\neq(N-1)/2}}^{(K_N(2l+1)-1)/2}\big(\beta_{2l+1,2k+1}(\beta_{n-2l-1,|N-2k-1|}+\beta_{n-2l,N+2k+1})\big)\Big).
\end{aligned}
\tag{53}
$$

Finally, for $\beta_{n,j}$, the result is zero whenever n and j have different parity. One must also consider that $\beta_{n,N\pm1}$ will depend on b_{n-1}, which is unknown. In practice, one can ignore the terms dependent on b_{n-1} at first, as this returns $\overline{\beta}$, which is all that is needed to eventually determine b_{n-1}. Once b_{n-1} is known, the previously ignored terms can be added to $\overline{\beta}$ to return the true value of the coefficient. We begin by presenting the formulas for even N. If n and j are both even, then

$$
\begin{aligned}
\beta_{n,j}=\frac{1}{\widehat{\mathcal{R}}(j)}\Bigg(&(1-\delta_{j,2})\beta_{n-1,j-1}+\beta_{n-1,j+1}+\sum_{l=1}^{\frac{n-2}{2}}\Big(s_{n-2l}\beta_{2l,j}\\
&+b_{2l}((1-\delta_{j,2N})\beta_{n-2l,|N-j|}+\beta_{n-2l,N+j})+\frac{1}{2}\delta_{j,2N}b_{2l}b_{n-2l}\\
&+\frac{1}{2}\sum_{\substack{k=1\\k\neq N/2}}^{K_N(2l)/2}\big(\beta_{2l,2k}((1-\delta_{2k,j}-\delta_{|2k-j|,N})\beta_{n-2l,|2k-j|}+(1-\delta_{2k+j,N})\beta_{n-2l,2k+j})\big)\\
&+\frac{1}{2}(1-\delta_{2l,n-2})\sum_{k=1}^{(K_N(2l+1)-1)/2}\big(\beta_{2l+1,2k+1}((1-\delta_{|2k+1-j|,1})\beta_{n-2l-1,|2k+1-j|}\\
&+\beta_{n-2l-1,2k+1+j})\big)\Big)\Bigg)
\end{aligned}
\tag{54}
$$

and for n and j both odd

$$
\beta_{n,j}=\frac{1}{\widehat{\mathcal{R}}(j)}\Bigg((\delta_{j,N-1}+\delta_{j,N+1})b_{n-1}+(1-\delta_{j,N+1})\beta_{n-1,j-1}+(1-\delta_{j,N-1})\beta_{n-1,j+1}
$$

$$+\sum_{l=1}^{\frac{n-3}{2}}\Big(s_{n-2l-1}\beta_{2l+1,j}+b_{2l}((1-\delta_{|N-j|,1})\beta_{n-2l,|N-j|}+\beta_{n-2l,N+j})$$

$$+\frac{1}{2}\sum_{\substack{k=1\\k\neq N/2}}^{K_N(2l)/2}\big(\beta_{2l,2k}((1-\delta_{|2k-j|,1})\beta_{n-2l,|2k-j|}+\beta_{n-2l,2k+j})\big)$$

$$+\frac{1}{2}\sum_{k=1}^{(K_N(2l+1)-1)/2}\big(\beta_{2l+1,2k+1}((1-\delta_{2k+1,j}-\delta_{|2k+1-j|,N})\beta_{n-2l-1,|2k+1-j|}$$

$$+(1-\delta_{2k+1+j,N})\beta_{n-2l-1,2k+1+j})\big)\Big)\Bigg). \tag{55}$$

For N odd, if n and j are both even,

$$\beta_{n,j}=\frac{1}{\widehat{\mathcal{R}}(j)}\Bigg((\delta_{j,N-1}+\delta_{j,N+1})b_{n-1}+(1-\delta_{j,2}-\delta_{j,N+1})\beta_{n-1,j-1}+(1-\delta_{j,N-1})$$

$$\beta_{n-1,j+1}+\sum_{l=1}^{\frac{n-2}{2}}\Bigg(s_{n-2l}\beta_{2l,j}+\frac{1}{2}\sum_{k=1}^{K_N(2l)/2}\big(\beta_{2l,2k}((1-\delta_{2k,j})\beta_{n-2l,|2k-j|}+\beta_{n-2l,2k+j})\big)$$

$$+(1-\delta_{2l,n-2})\Big[b_{2l+1}((1-\delta_{|N-j|,1}-\delta_{j,2N})\beta_{n-2l-1,|N-j|}+\beta_{n-2l-1,N+j})$$

$$+\frac{1}{2}\delta_{j,2N}b_{2l+1}b_{n-2l-1}+\frac{1}{2}\sum_{\substack{k=1\\k\neq(N-1)/2}}^{(K_N(2l+1)-1)/2}\big(\beta_{2l+1,2k+1}((1-\delta_{|2k+1-j|,1}-\delta_{|2k+1-j|,N})$$

$$\beta_{n-2l-1,|2k+1-j|}+(1-\delta_{2k+1+j,N})\beta_{n-2l-1,2k+1+j})\big)\Big]\Bigg)\Bigg), \tag{56}$$

and if n and j both odd

$$\beta_{n,j}=\frac{1}{\widehat{\mathcal{R}}(j)}\Bigg(\beta_{n-1,j-1}+\beta_{n-1,j+1}+\sum_{l=1}^{\frac{n-3}{2}}\Big(s_{n-2l-1}\beta_{2l+1,j}+b_{2l+1}(\beta_{n-2l-1,|j-N|}+\beta_{n-2l-1,j+N})$$

$$+\frac{1}{2}\sum_{k=1}^{K_N(2l)/2}\big(\beta_{2l,2k}((1-\delta_{|2k-j|,1}-\delta_{|2k-j|,N})\beta_{n-2l,|2k-j|}+(1-\delta_{2k+j,N})\beta_{n-2l,2k+j})\big)$$

$$+\frac{1}{2}\sum_{\substack{k=1\\k\neq(N-1)/2}}^{(K_N(2l+1)-1)/2}\big(\beta_{2l+1,2k+1}((1-\delta_{2k+1,j})\beta_{n-2l-1,|2k+1-j|}+\beta_{n-2l-1,2k+1+j})\big)\Big]\Big)\Bigg). \tag{57}$$

Maple code which finds the solution by executing the algorithm presented in Sect. 2 and python code which finds the solution through the recursive relations presented above can both be found at https://github.com/raylanger/WiltonRipples.

Appendix B: Proof of Theorems

B.1 Proof of Theorem 1

Proof For the proofs of Theorems 1 and 2, it is more convenient to use notation which is consistent for the coefficients of the resonant mode, wave speed correction, and the non-resonant modes. That is, instead of b_n, s_n, and $\beta_{n,k}$, we use $\gamma_{n,k}$, where in general $\gamma_{n,k} = \beta_{n,k}$, except for $\gamma_{n,N} = b_n$ and $\gamma_{n,0} = s_n$. Using this notation, the expansion of the solution in the small parameter ϵ is

$$f = \sum_{n=1}^{\infty} \epsilon^n \sum_{k=1}^{\infty} \gamma_{n,k}\phi_k = \epsilon f_1 + \sum_{n=2}^{\infty} \epsilon^n \sum_{k=2}^{\infty} \gamma_{n,k}\phi_k \tag{58}$$

$$c = c_0 + s = c_0 + \sum_{n=1}^{\infty} \epsilon^n s_n = c_0 + \sum_{n=1}^{\infty} \epsilon^n \gamma_{n,0}\phi_0. \tag{59}$$

Note that for all $n > 1$, we have excluded the first mode, equivalent to setting $\gamma_{n,1} = 0$, and that we have used the fact that the wave speed is a constant term to express it as the coefficient of the zeroth mode, since $\phi_0 = 1$. Substituting this into (13) allows it to be rewritten as

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \epsilon^{l+m} \sum_{a=1}^{\infty}\sum_{b=0}^{\infty} \gamma_{l,a}\gamma_{m,b}\phi_a\phi_b = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \epsilon^n \left(\hat{\mathcal{L}}(i) - c_0\right) \gamma_{n,i}\phi_i,$$

where the sums over l and a produce f, and the sum over m and b produce $s + f$, hence the addition of the zeroth mode. The right-hand side denotes $\mathcal{R}[f]$. We proceed by isolating terms at each order by ensuring $l + m = n$, or $l = n - m$, and using identity (12), the above becomes

$$\sum_{m=1}^{n-1}\sum_{a=1}^{\infty}\sum_{b=0}^{\infty} \gamma_{m,a}\gamma_{n-m,b}\phi_a\phi_b = \sum_{i=1}^{\infty} \left(\hat{\mathcal{L}}(i) - c_0\right) \gamma_{n,i}\phi_i$$

$$\frac{1}{2}\sum_{m=1}^{n-1}\sum_{a=1}^{\infty}\sum_{b=0}^{\infty} \gamma_{m,a}\gamma_{n-m,b}\left(\phi_{a+b} + \phi_{|a-b|}\right) = \sum_{i=1}^{\infty} \left(\hat{\mathcal{L}}(i) - c_0\right) \gamma_{n,i}\phi_i, \tag{60}$$

where it is now clear that the right-hand side contains all n-th order coefficients, and the left-hand side contains all lower order coefficients.

The following is a proof by induction. We begin with the base case obtained by solving up to the third order, shown from the beginning of Sect. 4 up to the solution of Eqs. (35) and (36). For $N \geq 4$, this produces $\gamma_{1,N} = 0$, $\gamma_{1,0} = 0$, $\gamma_{2,0} = -1/2\mathcal{R}(2)$, and $\gamma_{2,2} = 1/2\mathcal{R}(2)$. All other coefficients up to second order are known to be zero, except $\gamma_{2,N}$. The non-resonant coefficients at third order would also be known, though that is irrelevant to the following proof. As expected, all coefficients satisfy Theorem 1. After the proof, we will remark on the special case of $N = 3$.

Like any induction proof, we assume that Theorem 1 is true up to some order $n - 1$, and wish to show that it continues to be true after solving at order n. As before, the unknowns at order n are $\gamma_{n,k}$ for $k \neq 0$, 1, or N, $\gamma_{n-1,0}$, and $\gamma_{n-2,N}$. The reasoning behind this will be discussed shortly.

To solve (60), we equate the coefficients of ϕ_i. On the left-hand side of (60), this is done by setting $a+b=i \to b=i-a$, which is valid for $a \leq i$, and $|a-b|=i \to b=a+i$ (always valid) and $b=a-i$ (valid for $a \geq i$). Thus, we can rewrite the above to simply obtain the coefficients of ϕ_i as

$$\frac{1}{2}\sum_{m=1}^{n-1}\sum_{a=1}^{\infty}\gamma_{m,a}\left(\gamma_{n-m,a+i}+(1+\delta_{a,i})\gamma_{n-m,|a-i|}\right)=\left(\hat{\mathcal{L}}(i)-c_0\right)\gamma_{n,i}, \qquad (61)$$

where $\delta_{a,i}$ is used because for $a=i$, both $b=i-a$ and $b=a-i$ are valid, in which case $b=0$. In general, this is analogous to (19), though in the case of $i=1$ or $i=N$, we find the two solvability conditions, (32) and (33), respectively.

For $i \neq 1$ or N, the right-hand side of (61) is non-zero, and so $\gamma_{n,i}$ is the unknown in this equation. The key observation is that m and $n-m$ will have the same parities if n is even, but different parities if n is odd. Let us first consider n to be even, and solely the term $\gamma_{m,a}\gamma_{n-m,a+i}$. Since the orders m and $n-m$ have the same parity, the modes a and $a+i$ must also share that parity, or one of the coefficients will necessarily be zero, according to Theorem 1. This can only happen if i is even. Similar logic may be applied to $\gamma_{m,a}\gamma_{n-m,|a-i|}$, which means that if i is odd, the left-hand side of (61) will be zero, proving that $\gamma_{n,i}$ is zero for n even, i odd. The argument is nearly identical when n is odd, with the exception that now that m and $n-m$ have different parity, so too must a and $a+i$ (or a and $|a-i|$), requiring instead that i be odd, now proving that $\gamma_{n,i}$ is zero for n odd, i even.

Recall that the above argument is only sufficient for $i \neq 1$ or N. At these values of i, $\widehat{\mathcal{L}}(i)-c_0=0$, meaning that $\gamma_{n,i}$ does not appear on the right-hand side. Thus, we assume that $\gamma_{n,1}=0$, and use these equations to solve for the resonant coefficient and wave speed correction at previous orders. Furthermore, we must consider the resonant coefficient from two orders previous, as in (61) $\gamma_{n-1,N}$ is paired with $\gamma_{1,N\pm 1}$ for $i=1$ and $\gamma_{1,N\pm N}$ for $i=N$, all of which are zero according to the base case.

Once this is known, the argument is largely the same. It is still true that if n and i have different parities, all terms on the left-hand side of (61) will be zero, outside of those containing the unknowns $\gamma_{n-1,0}$ and $\gamma_{n-2,N}$. The wave speed correction is solved for using $i=1$, shown in (41). As i is odd, we will find $\gamma_{n-1,0}=0$ whenever n is even, which is when $n-1$ is odd. Likewise, the resonant coefficient is solved for using $i=N$, as seen in (43), and thus we find $\gamma_{n-2,N}=0$ whenever n and N have different parity, which coincides with when $n-2$ and N have different parity.

Thus, we have shown that if Theorem 1 is true up to some order $n-1$, it will continue to be true at order n. Alongside the base case, this proves that Theorem 1 holds for all orders. □

Remark The above proof was specific to $N \geq 4$. For $N=3$, the base case is changed which affects how the special cases $i=1$ and $i=N=3$ are treated. Solving up to third order for $N=3$ produces non-zero $\gamma_{1,3}$, $\gamma_{2,4}$ and $\gamma_{2,6}$, in addition to the coefficients which were non-zero in the base case for $N=4$, as seen at the beginning of Sect. 3. The result is that the solvability conditions obtained from the special cases produce coupled equations for the wave speed correction and resonant coefficient, as given by the matrix in (28). As

1 and 3 are both odd, we find that both s_{n-1} and b_{n-2} are zero whenever n is even, in accordance with Theorem 1.

B.2 Proof of Theorem 2

Proof For this proof, it is useful to consider an equivalent definition of Theorem 2: If n satisfies $k(N-2) \leq n < (k+1)(N-2)$ for some non-negative integer k, then the highest mode $K_N(n)$ found at $\mathcal{O}(\epsilon^n)$ is $n+2k$. Each increase of 2 in $K_N(n)$ relative to n will henceforth be referred to as a **boost**, and the goal of the proof is to show that the number of boosts at each order is equal to $\lfloor n/(N-2) \rfloor$.

The highest mode solved for at $\mathcal{O}(\epsilon^n)$ will be the highest mode ϕ_{a+b} in (61) for which $\gamma_{m,a}$ and $\gamma_{n-m,b}$ are both non-zero, for all choices of $m < n$. To produce the largest possible sum, a and b must necessarily be the highest modes at their respective orders, and so we may define the following formula

$$K_N^*(n) = \max\left(\{K_N(m) + K_N(n-m) \quad \forall\, 1 \leq m \leq n-1\}\right), \tag{62}$$

where $K_N^*(n)$ denotes the highest mode *solved for* at $\mathcal{O}(\epsilon^n)$, as opposed to $K_N(n)$, which denotes the highest mode *found*. There is an important distinction between *solved for* at $\mathcal{O}(\epsilon^n)$ and *found* at $\mathcal{O}(\epsilon^n)$. Typically, they are equivalent, as for most modes we solve for their coefficients at the current order. So, if a mode is solved for at some order, it is also found at that order. For only one mode is this not true, and that is the resonant mode ϕ_N, as we solve for $\gamma_{n-2,N}$ at $\mathcal{O}(\epsilon^n)$, as discussed in Theorem 1. $K_N^*(n)$ is still very useful as it always provides the highest *known* mode at $\mathcal{O}(\epsilon^n)$ after solving at that order.

Because the resonant mode is a special case, we must split the proof into two parts. The first is determining at what order the resonant mode first appears. Once this is known, the solution up to that order serves as a base case for the second part of the proof, an induction proof showing that Theorem 2 holds for all higher orders.

We will show that the resonant mode first appears at $\mathcal{O}(\epsilon^{N-2})$, which is also the first instance of a boost. This requires that we solve up to $\mathcal{O}(\epsilon^N)$, and in doing so, we will also show that for all $n < N-2$, the highest mode present at $\mathcal{O}(\epsilon^n)$ is ϕ_n. The proof of this follows and serves as a nice introduction to the rest of the proof. As with Theorem 1, we provide the proof for $N \geq 4$, and remark on the case of $N = 3$ at the end.

The initial condition $f_1 = \phi_1$ serves as the base case for an induction proof showing that $K_N(n) = n$ for $n < N-2$. From (62), it is clear that if $K_N(m) = m \;\forall\; m < n$, then $K_N^*(n) = m + (n-m) = n$. Alongside the initial condition, this shows that up to $\mathcal{O}(\epsilon^{N-3})$, the highest mode is the same as the order.

We proceed carefully to $\mathcal{O}(\epsilon^{N-2})$. It is also true that $K_N^*(N-2) = N-2$, and thus after solving at that order the highest known mode is ϕ_{N-2}. However, $\gamma_{N-2,N}$ remains an unknown. Thus, we tentatively set $K_N(N-2) = N-2$ to be used for the subsequent orders, but note that this may be subject to change should $\gamma_{N-2,N}$ be non-zero. Using $K_N(N-2) = N-2$, we find $K_N^*(N-1) = N-1$, but similarly $\gamma_{N-1,N+1}$ is still unknown as it depends on $\gamma_{N-2,N}$, as seen in (40). We again set $K_N(N-1) = N-1$ while acknowledging that this is not yet certain. Proceeding to $\mathcal{O}(\epsilon^N)$, we find $K_N^*(N) = N$, meaning that at this order we solve for the resonant mode. This produces a non-zero $\gamma_{N-2,N}$, the first instance of a boost, as well as a non-zero $\gamma_{N-1,N+1}$. While this means that we must retroactively change $K_N(N-2)$ to N, $K_N(N-1)$ to $N+1$, and $K_N(N)$ to

$N+2$, it is still true that $\mathcal{O}(\epsilon^N)$ is the first order at which we solve for the resonant mode, which is the important part of this proof.

Now that we have shown that the resonant mode always first appears at $\mathcal{O}(\epsilon^{N-2})$, we may proceed to the rest of the proof of Theorem 2. Recall that we have also shown that for all orders $m < N-2$, $K_N(m) = m$, in accordance with Theorem 2. We aim to show that the number of boosts only increases at orders which are multiples of $N-2$, and otherwise the highest mode increases by only 1 with the order. To do so, we show that all orders of the form $\mathcal{O}(\epsilon^{k(N-2)+p})$, with k and $p \in \mathbb{Z}$ and $0 \le p < N-2$, have k boosts.

Now that we are only considering orders $n > N-2$, $K_N^*(n)$ will always be greater than N, and thus we may use $K_N^*(n) = K_N(n)$. To simplify, we let m in (62) take on the form

$$m = j(N-2) + q, \quad j \in \{0, 1, 2 \ldots k\},\ q \in \{0, 1, 2 \ldots N-3\}, \tag{63}$$

such that the corresponding order always has j boosts. Substituting (63) and $n = k(N-2) + p$ into $n - m$, we obtain

$$k(N-2) - m + p = (k-j)(N-2) + p - q. \tag{64}$$

The number of boosts at this order is less clear, as it depends on the values of p and q. Substituting (63) and (64) into (44) produces

$$K_N(m) = j(N-2) + q + 2\left\lfloor \frac{j(N-2)+q}{N-2} \right\rfloor = j(N-2) + q + 2j \tag{65}$$

and

$$K_N\big(k(N-2) - m + p\big) = (k-j)(N-2) + p - q + 2\left\lfloor \frac{(k-j)(N-2)+p-q}{N-2} \right\rfloor. \tag{66}$$

Finally, substitution of (65) and (66) into (62) produces

$$K_N^*\big(k(N-2)+p\big) = \max\left(k(N-2) + p + 2j + 2\left\lfloor \frac{(k-j)(N-2)+p-q}{N-2} \right\rfloor\right), \tag{67}$$

with the maximum being taken over all values of j and q. The j terms will cancel once a value of q is chosen, so we simply must maximize the value of the floor function, which is achieved by choosing $q \le p$. Thus, we may express the highest mode

$$K_N^*\big(k(N-2)+p\big) = K_N\big(k(N-2)+p\big) = k(N-2) + p + 2k, \tag{68}$$

which evidently has k boosts.

Note that if $p = 0$, that is, if the order is a multiple of $N-2$, then only $q = 0$ satisfies $q \le p$. As p increases, the number of q which satisfy that inequality increases, but this still only leads to k boosts, and as such increasing p by 1 only increases the highest mode by 1. Additionally, if we let $p = -1$, then no values of q satisfy $q \leq p$, and thus that order will have $k - 1$ boosts, showing that the number of boosts does increment at orders which are exactly multiples of $N - 2$. This completes the proof of Theorem 2. □

Remark For $N = 3$, we begin with $f_1 = \phi_1 + b_1\phi_3$, meaning that the preliminary step of determining when the resonant mode first appears is unnecessary. Furthermore, as $N - 2 = 1$, we expect the number of boosts to increment at every order, and thus the highest mode should increase by 3 every time. This can simply be shown through an induction proof, using $K_3(1) = 3$ as a base case. If we assume that $K_3(m) = 3m \; \forall \; m < n$, then (62) returns $K_3^*(n) = K_3(n) = 3n$. This is sufficient to show that for $N = 3$, the highest mode is always 3 times the order, in accordance with Theorem 2.

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