# Using Alpha Shapes to Approximate Signal Strength Based Positioning Performance

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*Abstract*—**Received signal strength (RSS) is a common tool for locating a transmitter via a sensor network. It is considered common knowledge that RSS performs well if the transmitter is inside the convex hull of the sensor network, and poorly otherwise. However, this positioning rule of thumb is binary, vague, and not always accurate. In this letter, we consider the alpha shape, which is a generalization of the concept of a convex hull. We show that the alpha shape gives a more accurate estimate of localization performance than the convex hull, and we show how to use it to obtain a quasi-continuous and quantitative (rather than binary and qualitative) estimate of a sensor network's performance. We then compare this rule of thumb to more detailed, computationally intensive performance limits generated by the Cramer–Rao lower bound, with very good agreement.**

*Index Terms*—**Alpha hull, alpha shape, Cramer–Rao lower bound, received signal strength.**

## I. Introduction

Position awareness is important in applications such as law enforcement, military reconnaissance, emergency response, location-based billing, resource allocation and tracking, and electronic games. In source localization or geolocation, a Wireless Sensor Network (WSN) is used to locate the source of a Radio Frequency (RF) transmission [1], [2].

Geolocation may be accomplished through Received Signal Strength (RSS), Angle of Arrival (AOA), Time of Arrival (TOA), and/or Time Difference of Arrival (TDOA) measurements. Though each measurement type has its own merits, this paper focuses on RSS. RSS measurements can be obtained via cooperative or noncooperative approaches. In cooperative systems, such as cell phone handset geolocation by base stations, the digital signal can be demodulated and segregated from additive noise, so the reported RSS just contains the signal and not the noise. In noncooperative systems, such as locating emitters in a hostile environment, the RSS may be determined by integrating the observed Power Spectral Density (PSD), and as such it includes noise power as well.

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In this letter, we are interested in estimating the geolocation performance of a WSN based on its geometry. It is considered common knowledge that RSS performs well if the transmitter is inside the convex hull of the sensor network, and poorly otherwise [3]–[5]. However, this positioning rule of thumb is binary and qualitative, yielding only a "good or bad" judgment. Moreover, it is not always accurate, particularly if the WSN has an irregular shape and/or varying density of node placement. In this letter, we consider the alpha shape, which is a generalization of the concept of a convex hull. The key contribution of this paper is that we show that the alpha shape gives a more accurate estimate of localization performance than the convex hull, and we show how to use a set of alpha shapes to obtain a quasi-continuous and quantitative (rather than binary and qualitative) estimate of a sensor network's performance. The proposed approach provides discrimination of performance within the network's convex hull, which is not provided by the binary rule of thumb. That binary rule of thumb is the baseline that we are comparing to, as it is the only existing rule of thumb in the literature that we are aware of. The "ground truth" that we compare to is the performance bound determined by the Cramer–Rao Lower Bound (CRLB). However, the CRLB is impossible to estimate visually, which is the motivation for the proposed rule of thumb.

Often, the position estimation in [2]–[5] involves cooperatively estimating many node locations based on a small set of anchor nodes. However, this is often accomplished iteratively, by using the anchor nodes to determine the location of a nonanchor node, then adding it to the list of anchor nodes. As such, the process is mathematically similar to that of using a WSN to locate a single transmitter. Thus, the binary rule of thumb can be (and frequently is) applied to either problem.

Most papers on RSS-based positioning largely ignore the practical effects of range limitations. Since sensor ranges are limited in practice, in this paper we will use the RSS model developed in [6], which incorporates range limits into RSS by comparing the RSS value from the standard model to the background noise power. However, the results in this paper can be applied to the standard model as well, since the range limits can be removed by setting the background noise power to zero.

### A. Related Work

In [7], alpha shapes were used to estimate the coverage area of a WSN. However, their goal was to determine a binary approximation of the coverage area such that an event occurring within the coverage area was within range of at least one sensor, and thus could be detected. In contrast, we are interested in position estimation performance, and we seek a quasi-continuous (rather than binary) performance estimate. In [7], there was no discussion of how to choose the shape parameter $\alpha$ (or even what value of $\alpha$ was used in the simulations); whereas we relate the value of $\alpha$ to the expected position error, to quantify the performance estimate. Moreover, [7] required all sensors to exchange local information and report to a central node before the coverage area is estimated, whereas our approach is strictly geometric. In [8], the WSN determined the alpha shape in a distributed fashion, which, coupled with our results, may enable a node to determine how important it is to the localization process.

Optimal receiver geometries were examined in [9]–[11]. The results therein can be used to determine the optimal geometries if the sensors can be placed at will. In contrast, our work focuses on quickly evaluating a geometry that may or may not be optimal. If there are constraints on where the sensors may be placed, the optimal geometry may not be achievable, and our results can be used to quickly suggest alternate geometries that still perform well.

*Notation:* $(\cdot)^T$ denotes matrix transpose. Matrices and vectors are upper and lowercase boldface letters, respectively.

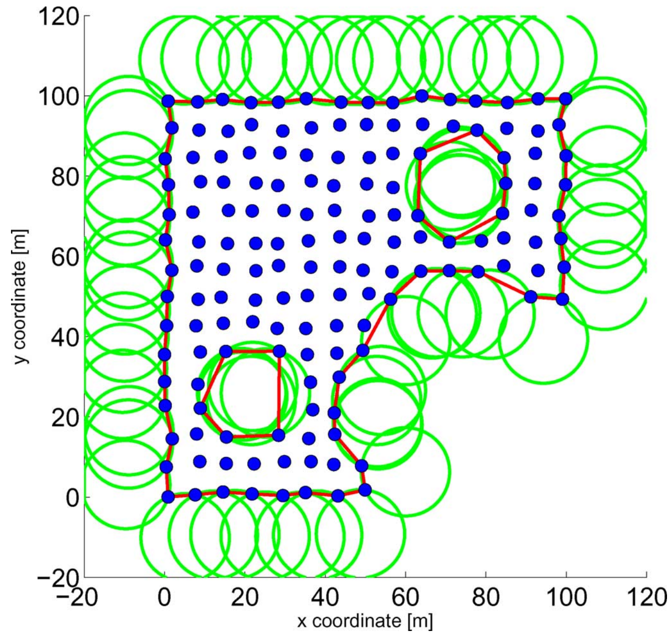


Fig. 1. Example alpha shape computation in a WSN. The radius $R$ for the green circles was 11 m. The blue dots are node locations and the red lines indicate the boundary of the alpha shape. Note the two "holes" within the shape where the sensor density is low.

## II. Alpha Shapes

Consider a set of points (known sensor positions in our case) $\mathcal{S} = \{(x_s, y_s),\ s = 1, 2, \ldots, N\}$. The alpha shape of $\mathcal{S}$ is a uniquely defined region that approximates its intuitive "shape" [12], [13]. Formally, for negative real $\alpha$, the *alpha hull* is the intersection of closed complements of discs of radius $R = -1/\alpha$ that each completely contain $\mathcal{S}$. The *alpha shape* is the straight line graph formed by connecting points of $\mathcal{S}$ that are neighbors on the boundary of the alpha hull, essentially replacing concave arcs of the alpha hull by lines. An example is shown in Fig. 1. The alpha shape can be described qualitatively by rolling a hoop of radius $R$ around the edges of the point cloud. If the hoop touches two points and there are no additional points in the hoop's interior, then the line segment containing the two points is on the boundary of the alpha shape.

*Definitions:* The *Delaunay triangulation* is the set of triangles (simplexes) with vertices in $\mathcal{S}$ such that no triangle's circumcircle contains any point in $\mathcal{S}$; the alpha shape is a subset of the Delaunay triangulation of $\mathcal{S}$. The *alpha complex* is the set of all simplexes of the Delaunay triangulation that are in the interior of the alpha shape; this is actually the region that will be shaded in Section IV, and its boundary is the alpha shape. The *convex hull* of $\mathcal{S}$ is the smallest convex set containing $\mathcal{S}$; think of placing a rubber band around the points of $\mathcal{S}$. The convex hull is equal to the alpha shape generated by $R \to \infty$ (or any $R$ greater than half the largest intra-point separation).

The alpha complex can be constructed by examining each simplex in the Delaunay triangulation and accepting or rejecting it per the conditions in [12]. Computationally, in two dimensions the Delaunay triangulation takes $\mathcal{O}(N \log N)$ time, and its resulting size is $\mathcal{O}(N)$. Thus, computation of the alpha shape is $\mathcal{O}(N \log N)$. Moreover, if we are interested in a range of values of $R$ (which is the case in this paper), the Delaunay triangulation only needs to be computed once, and only the $\mathcal{O}(N)$ step of accepting or rejecting simplexes must be repeated for each choice of $R$. Furthermore, note that if $R_1 < R_2$, then the shape of $R_1$ is contained in that of $R_2$. Thus, by starting at a large $R$ and working downwards, the set of simplexes for each $R$ decreases continually, further limiting the complexity.

Since there are only a finite number of edge lengths in the Delaunay triangulation, the set of possible alpha shapes is discrete and finite. Thus, as we will see in Section IV, the use of alpha shapes for performance assessment is necessarily discrete in nature, but there are generally enough choices of $R$ leading to distinct alpha shapes to provide a quasi-continuous performance assessment.

## III. Cramer–Rao Lower Bound

We will judge the geolocation performance of a WSN by its CRLB, which provides an algorithm-independent performance bound on the position estimation problem. The CRLB will be considered the "ground truth" performance assessment, which we want to approximate. The RSS model and associated CRLB are reviewed in this section. We then compute the CRLB for a uniform geometry in order to relate the $R$ values of an alpha shape to an associated approximate position error, as a sort of calibration.

The transmitter is at an unknown position $(x_0, y_0)$, at a distance

$$d_s = \sqrt{(x_s - x_0)^2 + (y_s - y_0)^2} \tag{1}$$

from sensor $s$. Before considering fading or background noise, the RSS is modeled by

$$m_{o,s} = P_0 - \eta \overline{d}_s, \tag{2}$$

$$\overline{d}_s = 10 \log_{10}\left(\frac{d_s}{d_0}\right) \tag{3}$$

where $P_0$ is the transmitted power as measured at a short reference distance $d_0$ (typically 1 m), and $\eta$ is the path loss exponent (2 in free space, but as large as 4 in some environments). In [6], it was shown that if the RSS incorporates a background noise floor at a power level of $\tau$ (in dB), the distribution of the RSS can be approximated as

$$\mathbf{p} \sim \mathcal{N}\left(\mathbf{m}, \sigma^2 \mathbf{I}\right) \tag{4}$$

$$m_s = 10 \log_{10}\left(10^{m_{o,s}/10} + 10^{\tau/10}\right) \tag{5}$$

where $\sigma$ is the fading standard deviation in dB.

Often, the parameters $P_0$ and $\eta$ are unknown, so the vector of unknowns is $\mathbf{z} = [x_0, y_0, P_0, \eta]^T$. The $4 \times 4$ CRLB is

$$\mathbf{J} = \frac{1}{\sigma^2} \sum_{s=1}^{N} \gamma_s^2 \left(\nabla_{\mathbf{z}} m_{o,s}\right)\left(\nabla_{\mathbf{z}} m_{o,s}\right)^T \tag{6}$$

$$\nabla_{\mathbf{z}} m_{o,s} = \left[\frac{e_{\mathrm{dB}}\eta}{d_s^2}\left(x_s - x_0\right),\ \frac{e_{\mathrm{dB}}\eta}{d_s^2}\left(y_s - y_0\right),\ 1, -\overline{d}_s\right]^T$$

$$\gamma_s \triangleq \frac{10^{m_{o,s}/10}}{10^{m_{o,s}/10} + 10^{\tau/10}} \tag{7}$$

$$e_{\mathrm{dB}} \triangleq 10 \log_{10}(e) \approx 4.343. \tag{8}$$

The factor $\gamma_s$ is due to the noise floor, and can be omitted if the more traditional RSS model is preferred (equivalent to the limit $\tau \to -\infty$), though realistically this or some similar range-limiting factor should be imposed.

To compare computational complexity to that of the alpha shape, the CRLB requires $\mathcal{O}(10N)$ operations (mostly multiplies) if $P_0$ and $\eta$ are known, and $\mathcal{O}(20N)$ operations if they are unknown, *per test point*. These computations must all be repeated for each possible test point $(x_0, y_0)$ in the vicinity of the WSN. Thus, the CRLB complexity scales inversely with the product of the desired $x$ and $y$ resolutions, whereas the alpha shape complexity only scales with the number of selected values of $R$. Moreover, plotting the CRLB requires computing and rendering a contour plot, which is computationally expensive; whereas plotting alpha shapes simply involves plotting a set of triangular patches. As such, the set of alpha shapes can be computed much faster than the CRLB. More importantly, alpha shapes can be approximated visually or produced by physical construction, but the CRLB cannot; so the alpha shape concept can be used by designers to mentally evaluate candidate geometries.

In order to convert the bound into a scalar with units of meters, we define the *position error bound* as $\sqrt{\mathrm{CRLB}(x) + \mathrm{CRLB}(y)}$. In order to make a correspondence between the value of $R$ in an alpha shape and the estimated position error bound, consider the uniform sensor geometry shown in the inset of Fig. 2. This geometry consists of a near-infinite number of sensors (4172 in our code), regularly spaced at the vertices of a tessellation of equilateral triangles of side length $2R$. This side length was chosen because along the boundary of an alpha shape, the node spacing is about or slightly less than $2R$. For this uniform sensor spacing, the position error bound was numerically computed, leading to Fig. 2, whose plots can be represented by

$$\text{position error bound} \approx 0.156\, \frac{\sigma}{\eta}\, R \tag{9}$$

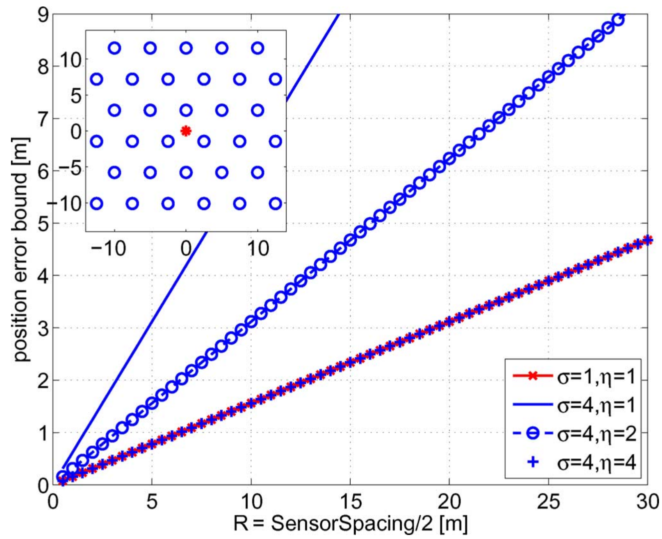


Fig. 2. Anticipated position error versus sensor spacing in a uniform WSN. A small window of the 4172-sensor uniform geometry is shown in the inset. The case "$\sigma = 4$, $\eta = 2$" is the subject of the example in Section IV.

with no dependence on $P_0$. Equation (9) was determined by numerically evaluating the CRLB for a variety of values of $\sigma$, $\eta$, and $R$. The factor of 0.156 was determined from the slope of the curves in Fig. 2 for $\sigma = \eta$.

To use (9), first construct a set of nested alpha shapes for a range of values of $R$; the Delaunay triangles for smaller values of $R$ will be a subset of those for larger values of $R$, as will be demonstrated in the example in Section IV. Using the smallest value of $R$ of all alpha shapes that include a given triangle, evaluate (9), yielding the estimated position error bound within that triangle. For an arbitrary geometry, (9) will only give a rough rule of thumb for the relationship between the error and $R$; however, that is still far better than the existing qualitative rule of thumb that says "inside the convex hull, good; outside, bad." Robustness of this approach will also be discussed after the example in the next section.

## IV. An Example

Figs. 3 and 4 show a numerical example for the 160-node WSN from Fig. 1. Fig. 3 shows the position error bound, and Fig. 4 shows the set of alpha shapes for 16 values of $R$, each labeled with its estimated position error bound. The alpha shapes were generated by overlaying the Delaunay triangles from smaller $R$ on top of those for larger $R$, which is easily accomplished by simply plotting all of them as $R$ decreases, and overwriting any existing triangles with new ones. The convex hull is shown by the outermost border in Fig. 4. The parameters were $P_0 = 20$ dBm, $\eta = 2$, $\sigma = 4$ dB, and $\tau = -10$ dB.

Regarding robustness of this approach, consider the case wherein the node positions are only approximately known, say to within $e$ meters. The line segment between any two nodes (and thus the edge lengths of any Delaunay triangle) are thus known to within $2e$ meters. If we consider the sizes of rings that we can or can't roll between these nodes, the radius of the ring that just barely fits is known to within $e$ meters. By (9), if $\sigma = 4$ and $\eta = 2$, the uncertainty in the position error is bounded by about $0.3 \cdot e$ meters. For example, if the node

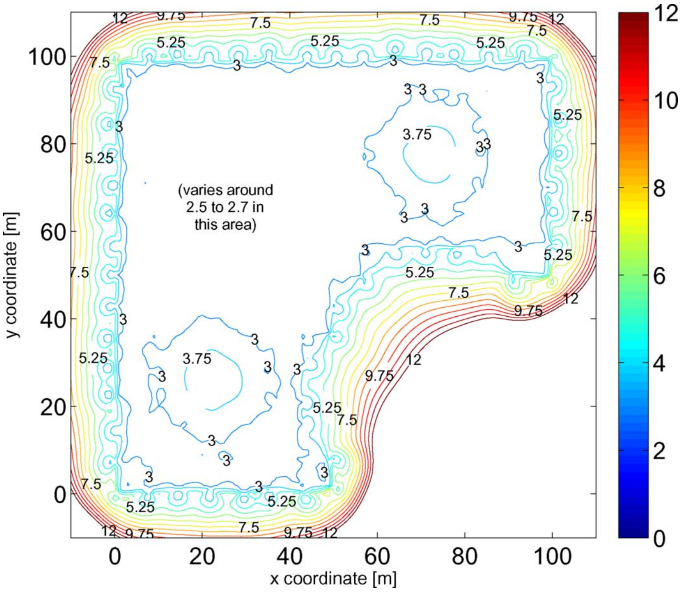


Fig. 3. CRLB on RSS-based positioning in a WSN. The colorbar indicates the position error in meters. For legibility, the contours were truncated above 12 m.

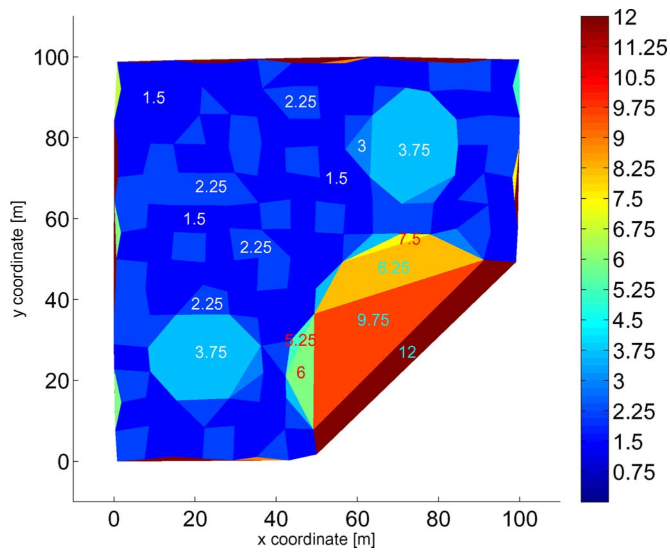


Fig. 4. Various alpha shapes of a WSN. The outermost alpha shape is also the convex hull. The numbers and colorbar indicate the estimated position error in meters, calculated by inserting the $R$ from each triangle into (9). The actual error is shown in Fig. 3.

position error is known to within 3 m, then the position error is known to within about 0.9 m.

## V. Conclusion

We have shown that the concept of the alpha shape provides a much better prediction of RSS-based geolocation performance than the traditional rule of thumb relating to the convex hull of the sensor locations. Specifically, it provides better performance discrimination within the convex hull of the sensors. We used a uniform geometry to provide a quantitative correspondence between values of $R$ and the predicted error bound, as a form of calibration. The virtues of using the alpha shape as a performance predictor are that it has low computational complexity, it can be approximated visually, and it can be constructed physically (e.g., for tutorial or illustrative purposes). This visual estimation may be helpful for designers, allowing mental evaluation of different sensor geometries in the absence of detailed computations.

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