HOPS Short Course: Stability of Traveling Water Waves

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Abstract In this contribution we present Higher Order Perturbation of Surfaces (HOPS) methods as applied to the spectral stability problem for traveling water waves. The Transformed Field Expansion method (TFE) is used for both the traveling wave and its spectral data. The Lyapunov-Schmidt reductions for simple and repeated eigenvalues are compared. The asymptotics of modulational instabilities are discussed.

1 Introduction

The water wave stability problem has a rich history, with great strides made in the late sixties in the work of Benjamin and Feir [1] and in the ensuing development of Resonant Interaction Theory (RIT) [2, 3, 4, 5]. The predictions of RIT have since been leveraged heavily by numerical methods; the influential works of MacKay and Saffman [6] and McLean [7] led to a taxonomy of water wave instabilities based on RIT (Class I and Class II instabilities). The most recent review article is that of Dias & Kharif [8]; since the publication of this review a number of modern numerical stability studies have been conducted [9, 10, 11, 12, 13].

In these lecture notes, we explain how the spectral data of traveling water waves may be computed using a High Order Perturbation of Surfaces (HOPS) approach, which numerically computes the coefficients in amplitude-based series expansions [14]. For the water wave problem, a crucial aspect of any numerical approach is the method used to handle the unknown fluid domain. Just as in the traveling waves lecture of this short course, numerical results will be presented from the Transformed Field Expansion (TFE) method, whose development for the spectral stability problem appears in [13, 15, 16, 17].

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The TFE method computes the spectral data as a series in wave slope/amplitude, and thus relies on analyticity of the spectral data in amplitude. A large number of studies of the spectrum have been made which do not make such an assumption [9, 18, 19, 10]. On the other hand, it is known that the spectrum is analytic for all Bloch parameters at which eigenvalues are simple in the zero amplitude limit [20]. Numerically it has been observed that the spectrum is analytic in amplitude at Bloch parameters for which there are eigenvalue collisions, but that the disc of analyticity is discontinuous in Bloch parameter. This discontinuity in radius is due to modulational instabilities, as explained in [21].

These notes begin by introducing the spectral stability problem for traveling water waves. Next, the TFE formulation is described, followed by leading-order asymptotics of the spectral data. In section 4, the perturbation series approach is discussed, followed by the Lyapunov-Schmidt reduction for triad collisions in section 5. Finally, we present instabilities due to the modulation of triad collisions in section 6.

2 Spectral Stability of Water Waves

The widely–accepted model for the motion of waves on the surface of a large body of water with constant surface tension, and in the absence viscosity, are the Euler equations

$$\phi_{xx}+\phi_{zz}=0, \qquad z<\varepsilon\eta, \tag{1a}$$
$$\phi_z=0, \qquad z=-H \tag{1b}$$
$$\eta_t+\varepsilon\eta_x\phi_x=\phi_z, \qquad z=\varepsilon\eta, \tag{1c}$$
$$\phi_t+\frac{\varepsilon}{2}\left(\phi_x^2+\phi_z^2\right)+\eta-\sigma\left(\frac{\eta_{xx}}{(1+\varepsilon^2\eta_x^2)^{3/2}}\right)=0, \qquad z=\varepsilon\eta, \tag{1d}$$

where η is the free–surface displacement and ϕ is the velocity potential. These equations describe the motion of an inviscid, incompressible fluid undergoing an irrotational motion. System (1) has been nondimensionalized as in [22, 15]. We assume that the wave slope, $\varepsilon = A/L$ is small (A is a typical amplitude and L, the characteristic horizontal length, is chosen in the non–dimensionalization so that the waves have spatial period 2π). Also, the vertical dimension has been non-dimensionalized using the wavelength, so the quantity H is non-dimensional ($H = h/L$). For simplicity sake, in these notes we will consider the deep water limit $H \to \infty$. To compute traveling waves, the time dependence of (1) is prescribed, using the ansatz

$$\eta(x,t)=\eta(x+ct) \qquad \text{and} \qquad \phi(x,z,t)=\phi(x+ct,z).$$

In the spectral stability problem, solutions to the traveling waves problem, $\bar{\eta},\bar{\phi},c$, are assumed to be known. These waves are then perturbed a small amount, δ, and linearized by substituting

$$\eta(x,t) = \bar{\eta}(x-ct) + \delta\zeta(x-ct)e^{\lambda t}, \qquad \phi(x,z,t) = \bar{\phi}(x-ct,z) + \delta v(x-ct,z)e^{\lambda t},$$

and neglecting quadratic powers of δ. The resulting problem will be a non-constant coefficient, generalized spectral problem, still on an unknown domain. For amplitude expansion based methods, an effective way to handle the domain is with the Transformed Field Expansion [23]. The TFE method is both spectrally accurate and numerically stable; see the the previous lecture in this short course or [24] for a comparison of other methods in the traveling wave problem.

2.1 Transformed Field Expansions

The Transformed Field Expansion method (TFE) solves for the field (perturbations of the velocity potential and free surface) via an amplitude expansion of the Euler equations (1) after two transformations. First, Laplace's equation is solved exactly below a prescribed depth, $z = -a$ (via the same operator presented in part II of this short course). Second, the domain above this depth is transformed to a strip by the simple change of variables,

$$z \to a\left(\frac{z-\varepsilon\eta}{a+\varepsilon\eta}\right),$$

after which the stability problem becomes

$$v_{xx} + v_{zz} = \tilde{F}, \qquad -a < z < 0, \tag{2a}$$
$$v_z - Tv = \tilde{J}, \qquad z = -a, \tag{2b}$$
$$\lambda\zeta + c\zeta_x - v_z = \tilde{Q} \qquad z = 0, \tag{2c}$$
$$\lambda v + cv_x + (1-\sigma\partial_x^2)\zeta = \tilde{R}, \qquad z = 0. \tag{2d}$$

The symbols $\tilde{F}, \tilde{J}, \tilde{Q}$ and $\tilde{R}$ contain all the non-constant coefficient terms, depending on $\bar{\eta}, \bar{\phi}, \zeta$, and u. This formulation yields stable, fast recursions in a Boundary Perturbation method, the formulation is discussed at length in [13, 15].

To consider the broadest class of perturbations, we will append Bloch boundary conditions to (2). If the traveling wave is of period $L = 2\pi$, the perturbations satisfy,

$$\zeta(x+2\pi) = \zeta(x)e^{2\pi ip}$$

with similar boundary condition for v, in which $p \in \mathbb{R}$ is the Bloch, or Floquet, parameter. This decomposes the continuous spectrum of the original problem (which has periodic coefficients) to a sets of discrete eigenvalues for each value of the Bloch parameter. After this decomposition, it makes sense to construct a perturbation expansion about the flat-state eigenvalues.

To compute the spectrum at fixed Bloch parameter, all variables are expanded as a series in amplitude, both the traveling wave data (as in the traveling wave lecture of this short course) and the spectral data,

$$v = \sum_{n=0}^{\infty} v_n \varepsilon^n, \qquad \zeta = \sum_{n=0}^{\infty} \zeta_n \varepsilon^n, \qquad \text{and} \qquad \lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n. \tag{3}$$

In the TFE formulation, this results in a sequence of linear problems

$$v_{n,xx} + v_{n,zz} = \tilde{F}_n(x,z), \qquad -a < z < 0, \tag{4a}$$
$$v_{n,z} - T v_n = \tilde{J}_n(x), \qquad z = -a, \tag{4b}$$
$$\lambda_0 \zeta_n + c_0 \zeta_{n,x} - v_{n,z} = \tilde{Q}_n(x) - \lambda_n \zeta_0, \qquad z = 0, \tag{4c}$$
$$\lambda_0 v_n + c_0 v_{n,x} + (1 - \sigma \partial_x^2)\zeta_n = \tilde{R}_n(x) - \lambda_n v_0, \qquad z = 0. \tag{4d}$$

The exact formula for the $\tilde{F}_n, \tilde{J}_n, \tilde{Q}_n$ and $\tilde{R}_n$ can be found in [13, 20], and we direct the motivated reader to the (tedious) details provided therein. These equations can be rapidly solved via Fourier collocation in the horizontal dimension, and an elliptic solver in the vertical dimension, for example the Chebychev-Tau method [15]. It is in this formulation that the spectrum has been calculated to all orders, about both simple and repeated eigenvalues, in deep water and finite depth, with and without surface tension [15, 16, 17].

3 Leading Order Behavior

In the flat water configuration, when $n = 0$, the right hand side of (4) vanishes, $\tilde{F}_0 = \tilde{J}_0 = \tilde{Q}_0 = \tilde{R}_0 = 0$. The resulting problem is exactly solvable, with eigenvalues

$$\lambda_0 = \pm i\omega(k_j) + ic_0 \cdot k_j, \tag{5}$$

where k_j are the wave numbers of the perturbation (including the Bloch parameter) and $\omega(k) = \sqrt{|k|(1+\sigma|k|^2)}$ is the dispersion relation of the potential flow equations (1). The leading order solution for the eigenfunctions depends on the multiplicity of the eigenvalue in question. We will discuss these solutions separately.

3.1 Simple eigenvalues

If the spectrum is simple, ignoring perturbations of the mean, so that $k_1 \neq 0$, the leading order eigenfunctions are

$$\begin{pmatrix} \zeta_0(x) \\ v_0(x,0) \end{pmatrix} = \beta_{0,1} \begin{pmatrix} 1 \\ \frac{\lambda_0 + ic_0 \cdot k_1}{|k_1|} \end{pmatrix} e^{ik_1 \cdot x} = \beta_{0,1} \upsilon_1 \tag{6}$$

Before one can compute (ζ_n, v_n), recall that the size of an eigenfunction is not a meaningful quantity. We choose to define the size of the eigenfunctions by the size of their Fourier coefficient at k_1, here $\hat{\zeta}(k_1) = 1$. Such a choice is necessary,

without which one cannot hope to uniquely compute an eigenfunction. This particular normalization of the eigenfunctions allows one to avoid including homogeneous solutions at later perturbation orders, simplifying the entire procedure.

Without collisions, the equations for λ_n are linear, and all λ_n are pure imaginary; simple eigenvalues do not lead to instability (within the radius of convergence of their series expansions). This observation motivated in-depth study of the radius of convergence of these expansions. Since the spectrum is simple almost everywhere (for almost all values of the Bond number and Bloch parameters), one might expect that the radius of convergence of these series could be used to detect instabilities, see [13]. Eigenvalue collisions however, even those of opposite Krein signature, do not always create instabilities. Next we consider the spectrum about resonant Bloch parameters, where eigenvalues collide.

3.2 Eigenvalue collisions

At Bloch parameters where the flat-state ($\varepsilon = 0$) contains an eigenvalue collision, $\lambda_0(k_1) = \lambda_0(k_2)$, implying

$$\pm\omega(k_1) \pm \omega(k_2) = (k_1 - k_2)c_0.$$

Including the broadest class of perturbations, $k_j = n_j + p$ with $p \in [0,1)$ and $n_j \in \mathbb{Z}$, this condition can be rewritten as

$$k_1 - k_2 = mk_0 \qquad \text{and} \qquad \omega(k_1) \pm \omega(k_2) = m\omega(k_0), \tag{7}$$

where $k_0 = 1$ is the frequency of the Stokes' wave, and $m = k_1 - k_2 \in \mathbb{Z}$. Equation (7), states that the perturbations are waves whose temporal and spatial frequencies resonant with m copies of the Stokes wave (for $m+2$ total waves). It is natural to label these resonances with the naming convention of Resonant Interaction Theory (RIT). In RIT, $m = 1$ is labeled a triad interaction, $m = 2$ is labeled a quartet, $m = 3$ is a quintet, etc. [3, 7, 6]. The leading order perturbations at repeated eigenvalues are superpositions of the eigenfunctions from the simple case,

$$\begin{pmatrix} \zeta_0(x) \\ v_0(x,0) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_0 + ic_0\cdot k_1}{|k_1|} \end{pmatrix} e^{ik_1\cdot x} + \beta_{0,2} \begin{pmatrix} 1 \\ \frac{\lambda_0 + ic_0\cdot k_2}{|k_2|} \end{pmatrix} e^{ik_2\cdot x} = \upsilon_1 + \beta_{0,2}\upsilon_2. \tag{8}$$

Just as for simple eigenvalues, the size of the eigenfunction is chosen so that $\hat{\zeta}(k_1) = 1$, later orders will not be supported at wavenumber k_1. From the perspective of the leading order problem, $\beta_{0,2}$ is free to take any value. At later orders, solvability conditions will allow only a discrete set of $\beta_{0,2}$, analogous to the Wilton ripple discussion earlier in this short course. The form of the solvability conditions, and the solution at later orders depends on the the type of eigenvalue collision which occurs at leading order, encoded by the value of m. All values of m are discussed in

[15]; for brevity, we present only the triad case, $m = 1$. First we will introduce the perturbation series in a simplified notation.

4 Eigenvalue perturbation

After Bloch decomposition, (2) can be thought of as a generalized eigenvalue problem,

$$(A - \lambda B)v = 0.$$

In this notation, (4) is recast

$$(A_0 - \lambda_0 B_0)v_n = \sum_{p=0}^{n-1} \left(A_{n-p} - \sum_{q=0}^{n-p} \lambda_q B_{n-p-q} \right) v_p \tag{9}$$

To construct the solution (v, λ), requires knowledge of the solutions of the adjoint of the leading order problem,

$$(A_0 - \lambda_0 B_0)^* \psi = 0. \tag{10}$$

Simple eigenvalues have a single non-trivial solution to (10), which we label ψ_1. At a generic eigenvalue collision (of only two eigenvalues), equation (10) has two solutions ψ_1 and ψ_2. For discussion purposes, we will choose to label the solutions based on the wavenumber (so that function ψ_j has spatial dependence proportionate to $e^{ik_j x}$). The eigenfunctions of the leading order problem

$$(A_0 - \lambda_0 B_0)v_0 = 0 \tag{11}$$

will be similarly labeled, so that $v_0 = \upsilon_1$, or in the case of an eigenvalue collision $v_0 = \upsilon_1 + \beta_{0,2}\upsilon_2$. As in the previous section, the eigenfunctions υ_j are labeled so that they are supported at wavenumber k_j.

At every order, n, solving equation (9) requires two steps. First, one must impose that the right hand side is orthogonal to the solutions of (10). Afterward, the equation is solvable, and one may invert the linear operator against its range. For simple eigenvalues, both steps are trivial. Solvability requires enforcing

$$\lambda_n = \frac{1}{\langle \psi_1, B_0 v_0 \rangle} \left\langle \psi_1, \sum_{p=0}^{n-1} A_{n-p} v_p - \sum_{q=0}^{n-1} \sum_{p=0}^{n-q-1} \lambda_q B_{n-p-q} v_p \right\rangle.$$

Given the terms in the series expansion for the spectral data, one may ask "to what extent is this series summable?" The answer informs on the values of ε for which a boundary perturbation method can compute the spectrum, i.e. the radius of the disc of analyticity of the spectrum. This radius may be computed using a standard convergence test on the terms in the Taylor series (3), or by computing the Padé interpolant and finding its smallest uncancelled pole. An example of the

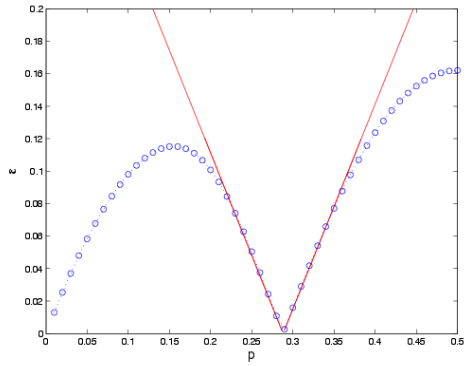

Fig. 1 The radius of convergence of the spectral data as a function of Bloch parameter is numerically computed (circles). A triad eigenvalue collision occurs at $(p,\varepsilon) \approx (0.29,0)$. An asymptotic prediction, equation (14), for the location of modulated instabilities, and thus loss of analyticity, is marked with the solid line.

numerically computed radius of convergence of this series as a function of Bloch parameter p is marked with circles in Figure 1.

5 Triad Instabilities

Triad instabilities arise about eigenvalue collisions $\lambda_0(k_1) = \lambda_0(k_2)$, where

$$k_1 - k_2 = \pm k_0 \qquad \text{and} \qquad \omega(k_1) \pm \omega(k_2) = \pm\omega(k_0)$$

in which k_0 is the Stokes wave, here $k_0 = 1$. Triad resonances then, are those where both the spatial and temporal frequencies of the perturbations differ by exactly those of the Stokes wave.

The boundary perturbation approach expands the traveling wave as a series, where the $O(\varepsilon^n)$ term is supported at wave numbers $|k| \leq n$. The traveling wave corrections (η_{n-j}, ϕ_{n-j}) occur as coefficients of the previous eigenfunctions (ζ_j, v_j) in the N^{th} right hand sides $(\tilde{F}_n, \tilde{J}_n, \tilde{Q}_n, \tilde{R}_n)$ of equation (4). In order to get non-trivial nonlinear solvability conditions, the coefficients of the wavenumber k_2 must appear in the solvability condition for k_1, thus there must be a wave interaction with $k_1 - k_2 = m$ copies of the Stokes wave. Wavenumber $k = m$ is first present in the traveling wave at $O(\varepsilon^m)$, thus for triads this first occurs at $O(\varepsilon)$. These first solvability conditions are

$$\begin{pmatrix} \langle \psi_1, (A_1 - \lambda_0 B_1 - \lambda_1 B_0) v_1 \rangle & \langle \psi_1, (A_1 - \lambda_0 B_1 - \lambda_1 B_0) v_2 \rangle \\ \langle \psi_2, (A_1 - \lambda_0 B_1 - \lambda_1 B_0) v_1 \rangle & \langle \psi_2, (A_1 - \lambda_0 B_1 - \lambda_1 B_0) v_2 \rangle \end{pmatrix} \begin{pmatrix} \beta_{0,1} \\ \beta_{0,2} \end{pmatrix} = 0. \quad (12)$$

For the water wave problem, a number of these inner products vanish. In our labeling, ψ_j and υ_j are supported at k_j, and since A_1 and B_1 are supported at wavenumber $k = \pm 1$,

$$\langle \psi_1, (A_1 - \lambda_0 B_1)\upsilon_1 \rangle = \langle \psi_1, \lambda_1 B_0 \upsilon_2 \rangle = \langle \psi_2, (A_1 - \lambda_0 B_1)\upsilon_2 \rangle = \langle \psi_2, \lambda_1 B_0 \upsilon_1 \rangle = 0.$$

An inner product can be non-zero only if the wave numbers of the functions in the inner product sum to zero, hence the connection to RIT. The leading order eigenvalue correction is

$$\lambda_1 = \pm \sqrt{\frac{\langle \psi_1, (A_1 - \lambda_0 B_1)\upsilon_2 \rangle \langle \psi_2, (A_1 - \lambda_0 B_1)\upsilon_1 \rangle}{\langle \psi_2, B_0 \upsilon_2 \rangle \langle \psi_1, B_0 \upsilon_1 \rangle}} = \pm \sqrt{\tau_{1,2}\tau_{2,1}}.$$

The $\tau_{i,j}$, defined implicitly above are

$$\tau_{i,j} = \frac{\langle \psi_i, (A_1 - \lambda_0 B_1)\upsilon_j \rangle}{\langle \psi_i, B_0 \upsilon_i \rangle}.$$

At this order, $\beta_{0,j}$ are determined as the null vectors in (12). For deep water two-dimensional Stokes waves, labeling $k_1 = k_2 + k_0$, these inner products evaluate to

$$\tau_{1,2} = \frac{1}{2}\left(\frac{i\omega(k_1)}{1+\sigma k_1^2} \left((1+\sigma)(|k_2| - k_2)\frac{\omega(k_2)}{c_0|k_2|} + \omega^2(k_2) + c_0^2 \right) - ik_1 \left(\frac{\omega(k_2)}{|k_2|}k_2 + c_0 \right) \right),$$
$$\tau_{2,1} = \frac{1}{2}\left(\frac{i\omega(k_2)}{1+\sigma |k_2|^2} \left((1+\sigma)(-|k_1| - k_1)\frac{\omega(k_1)}{c_0|k_1|} + \omega^2(k_1) + c_0^2 \right) - ik_2 \left(\frac{\omega(k_1)}{|k_1|}k_1 + c_0 \right) \right).$$

From the above formulae, we see that the $\tau_{i,j}$ are pure imaginary, and stability is determined by the sign of the product $\tau_{1,2}\tau_{2,1}$.

At all later orders, the corrections to the perturbations are decomposed into two parts

$$v_n = v_{n,p} + v_{n,h}$$

where the $v_{n,p}$ are particular solutions, chosen to be orthogonal to the null vectors υ_1 and υ_2. The $v_{n,h} = \beta_{n,2}\upsilon_2$ are the the homogeneous solutions of (9) at $n = 0$, whose coefficients $\beta_{n,2}$ are set by solvability (as are the λ_n). This decomposition was unnecessary for simple eigenvalues, as our choice of normalization of the eigenfunctions set $v_{n,h} = 0$ for $n > 0$. The general order, $n \geq 2$, corrections solve

$$(A_0 - \lambda_0 B_0)v_n = -\sum_{j=1}^{n} \left(A_j - \sum_{k=0}^{j} \lambda_k B_{j-k} \right) v_{n-j} \tag{13}$$

To solve for v_n in equation (13), one must first impose that the right hand side is in the range of the linear operator, $(A_0 + \lambda_0 B_0)$. At triads for $n \geq 2$ the resulting solvability conditions are linear,

$$\begin{pmatrix} \langle \psi_1, (A_1-\lambda_1 B_0-\lambda_0 B_1)\upsilon_2\rangle & \langle \psi_1, B_0 v_0\rangle \\ \langle \psi_2, (A_1-\lambda_1 B_0-\lambda_0 B_1)\upsilon_2\rangle & \langle \psi_2, B_0 v_0\rangle \end{pmatrix} \begin{pmatrix} \beta_{n-1,2} \\ \lambda_n \end{pmatrix} = -\begin{pmatrix} \left\langle \psi_1, (A_1-\lambda_1 B_0-\lambda_0 B_1) v_{n-1,p} \right\rangle \\ \left\langle \psi_2, (A_1-\lambda_1 B_0-\lambda_0 B_1) v_{n-1,p} \right\rangle \end{pmatrix}$$
$$+\begin{pmatrix} \left\langle \psi_1, \left(A_n-\sum_{j=0}^{n-1}\lambda_j B_{n-j}\right) v_0 \right\rangle \\ \left\langle \psi_2, \left(A_n-\sum_{j=0}^{n-1}\lambda_j B_{n-j}\right) v_0 \right\rangle \end{pmatrix} + \begin{pmatrix} \left\langle \psi_1, \sum_{j=2}^{n-1}\left(A_j-\sum_{k=0}^{j}\lambda_k B_{j-k}\right) v_{n-j} \right\rangle \\ \left\langle \psi_2, \sum_{j=2}^{n-1}\left(A_j-\sum_{k=0}^{j}\lambda_k B_{j-k}\right) v_{n-j} \right\rangle \end{pmatrix}.$$

Thus to compute the general term, one must impose (14), then solve (13) for $v_{n,p}$. The complete correction at order $O(\varepsilon^n)$, is not known until the solvability conditions are imposed at $O(\varepsilon^{n+1})$, at which point we have determined $v_{n,h}$.

The above series expansions, and similar expansions about quartets, quintets etc, compute the spectrum at any *fixed* values of the Bloch parameter. This approach works well both far from, and exactly at, resonant Bloch parameters. It does not perform well in the neighborhood of resonant Bloch parameters, the radius of convergence of the series shrinks as it approaches resonant configurations, see Figure 1. This is due to the change in form of the series at resonant Bloch parameters, where flat state eigenvalues collide, and the linear operator $(A_0-\lambda_0 B_0)$ has two dimensional kernel. To study the spectrum near, but not exactly at, resonant Bloch parameters, one must include the effect of modulation, by allowing the Bloch parameter to depend on amplitude.

6 Modulations of Triads

In this section we refer to modulational instabilities as those whose Bloch parameter depends on amplitude. Often the term modulational instability is used to refer only to the long-wave, Benjamin-Feir instability. Our use of the term modulation is consistent with its meaning in the Benjamin-Feir setting, as the expansion in Bloch parameter can be thought of as including the effects waves modulated by their sidebands. For brevity reasons, we will discuss only modulational instabilities of triads; for quartets and Benjamin-Feir see [21].

To compute the modulated spectrum, first notice that the operators A and B, and thus the eigenfunctions and eigenvalues (v,λ), are all functions of the Bloch parameter p. To compute modulational instabilities, we couple to equation (3), an amplitude expansion for the Bloch parameter

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots$$

This expansion introduces an extra unknown at each order in ε. Since the series was already solvable, the Bloch parameter corrections p_j cannot be determined by equation (9). What results instead is the functional dependence of the spectrum on the Bloch parameter corrections.

Triad instabilities arise at $O(\varepsilon)$; non-trivial modulation of these instabilities can also be recovered at this order. The modulational contribution (the variations of the operators A_j, B_j with respect to frequency) appear in the triad solvability conditions.

These modulations occur in the form of perturbations of the phase speed and group velocity. For a general triad, the $\tau_{i,j} \neq 0$, and the first nonzero correction to the flat-state spectrum is

$$\lambda_1 = -i\left(c_0 - \frac{c_g(k_1)+c_g(k_2)}{2}\right)p_1 \pm \frac{1}{2}\sqrt{\tau_{1,2}\tau_{2,1} - ((c_g(k_2)-c_g(k_1))p_1)^2},$$

where $c_g(k_j) = \omega_k(k_j)$ is the group velocity vector at wave number k_j.

Both τ_j are pure imaginary, so if $\tau_{2,1}\tau_{1,2} > 0$, then there is a band of p_1 where instabilities occur, which includes the non-modulated case $p_1 = 0$. Instabilities exist within the symmetric interval

$$|p_1| < \frac{\sqrt{\tau_{1,2}\tau_{2,1}}}{|c_g(k_1) - c_g(k_2)|}. \tag{14}$$

The boundaries of this interval are marked by the solid straight lines in Figure 1. In this figure, we see that the boundaries of the region where modulated instabilities occur predicts well the radius of convergence of a non-modulational expansion of the spectrum. On the other hand the largest triad instabilities are the non-modulational ones; the triads in the band of instabilities where λ_1 has the largest real part are at non-modulational, at $p_1 = 0$.

The effect of modulation can be considered in the absence of triads. If there is no triad interaction, then $\tau_{i,j} = 0$, and there can be no instability at $O(\varepsilon)$. There may be instability at later orders, with its scaling and character depending on the degree of the resonance. The cases of quartets and the four-eigenvalue collision Benjamin-Feir instability are discussed in detail in [21]. The lesson to be learned here is that non-modulational expansions of the spectra lose their analyticity at asymptotically small locations, which can be predicted using a modulational expansion of the spectral data.

In these notes, we focus on the leading order asymptotics of the spectrum. When implementing such an expansion to all orders a number of details become important. For example, one must consider convergence of the modulational expansion of the operators (in frequency space in addition to amplitude space). The effects of floating point cancellations become crucially important. The TFE method was derived to elegantly deal with such cancellations, alternatively one may use extended precision for intermediate computations - see [24]. The cost of such an expansion is also a factor, which can be significantly reduced by solving for the corrections recursively, see for example [22].

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