Confidence Intervals for the Median and Other Percentiles

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Executive Summary

This best practice explains an approach to construct confidence intervals for the median and other percentiles by walking through an example in JMP. When the distribution of a statistic for a population characteristic of interest is known, we can use the properties of this distribution to construct confidence intervals of that population characteristic. For example, if the population has a normal distribution, then the sample mean has a normal distribution and we use this information to construct confidence intervals of the population mean. The construction of confidence intervals for the median, or other percentiles, however, is not as straightforward.

Keywords: confidence interval, median, percentile, statistical inference

Introduction

Kensler and Cortes (2014) and Ortiz and Truett (2015) discuss the use and interpretation of confidence intervals (CIs) to draw conclusions about some characteristic of a population. These best practices provide examples of CIs for a population proportion and population mean, respectively. In this best practice, let us assume that our characteristic of interest is a continuous variable. If we know that the underlying distribution of this variable is normally distributed, we can use the techniques discussed by Ortiz and Truett (2015) to calculate a CI from a random sample of data from our population. However, what is the correct approach when the assumptions required for the CI do not apply?

If the assumptions of CIs for the mean do not hold for your data or the distribution of your population is unknown, it may be advantageous to estimate the median. There may also be cases where a percentile (for example the 75th or 95th percentile) may be of more interest than the center of the data. We can easily calculate an estimate of the population percentiles from a random sample (see below). However, this is a point estimate: a single value that estimates the population percentile. Rather than provide only a single value, we would like to also determine a confidence interval on the population percentile. This would provide us a realistic range of values for the percentile with a given degree of confidence. In this best practice, we demonstrate how to determine CIs of population percentiles, including the median. The technique is demonstrated using JMP (V.12). The appendix provides the mathematical details for those interested.

Definitions and Notation

We first introduce some definitions and notation to explain the method of constructing CIs for percentiles.

Percentile: The pth percentile (denoted x_p) is the value x of a population/random variable such that $\mathrm{P}(X \leq x) = p$. The pth (sample) percentile (denoted $\hat{x}_p$) is the value such that $100p\%$ of the sample is smaller than x. Equivalently, $100(1-p)\%$ of the data lies above x (Kvam and Vidakovic, 2007). The median, for example, is the 50th percentile. 50% of the population falls below the median and 50% lies above the median. The 75th percentile, $x_{0.75}$, is the value such that 75% of the population falls below $x_{0.75}$ and 25% lies above $x_{0.75}$.

Order Statistic: Let $X_1, X_2, \dots, X_n$ be a random, independent sample from a population. The sample can be ordered in an ascending order and denoted as $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ such that:

$$X_{(1)} < X_{(2)} < \cdots < X_{(n-1)} < X_{(n)}$$

where $X_{(i)}$ denotes the i^{th} largest value in the sample. So, for example, $X_{(1)}$ denotes the minimum and $X_{(n)}$ denotes the maximum. $X_{(i)}$ is called an **order statistic**. Order statistics are commonly used in nonparametric statistics, a field of statistics that does not rely on assumptions of the distribution of the population. (A side note: nonparametric statistics does not mean "assumption-free!") We can use order statistics to determine a confidence interval for the median of a population (or any other percentile). There are many theoretical properties regarding order statistics (see Kvam and Vidakovic, 2007 or Casella and Berger, 2002 for details).

Estimating Percentiles

For large samples, there is often a rank number r between 1 and the sample size n such that $X_{(r)} = x_p$. In other words, a value in the sample is the pth percentile if $p(n+1) = r$ (Kvam and Vidakovic, 2007). For example, a random sample of 5 observations has the values 4, 2, 7, 5, 9. Arranging this sample in ascending order gives us 2, 4, 5, 7, 9. The 50th percentile (the median) corresponds to the 3rd order statistic $X_{(3)} = 5$ since $0.5(5+1) = r = 3$. However, note that if we wish to estimate the 75th percentile in this way, there is not an integer r between 1 and n such that $0.75(5+1) = r$.

If $p(n+1)$ is not an integer, we can interpolate the percentile between $X_{(r)}$ and $X_{(r+1)}$, often done with software. For example, if the sample size is even, the median can be estimated as $M = \frac{X_{(n)} + X_{(n+1)}}{2}$. If your sample size is odd, the median can be estimated as $M = X_{\left(\frac{n+1}{2}\right)}$, as we saw above.

Finding the Confidence Limits Using JMP

The previous section explained how to estimate a percentile with a single value. The goal is to identify values $X_{(j)}$ and $X_{(k)}$ in the sample such that $\mathrm{P}\left(X_{(j)} \le x_p \le X_{(k)}\right) = 1 - \alpha$, where α denotes the probability of a type I error and $1 - \alpha$ denotes the confidence level. For example, $\mathrm{P}\left(X_{(j)} \le x_{0.50} \le X_{(k)}\right) = 0.95$ would provide us a 95% CI of the population median using values contained in the sample. Note how this approach is different compared to CIs for the mean and proportion discussed previously. Those approaches take on the general form of:

$$s = C_{(conf\ level,n)} s.e.(s),$$

where s is some statistic, C is a critical value based on the confidence level and sample size, and $s.e.(s)$ is the standard error of the statistic. This is a parametric approach, meaning it uses properties of the distribution of the statistic to determine the lower and upper confidence bounds. CIs for percentiles uses a nonparametric approach, which, as mentioned previously, does not use any information about the distribution of the statistic. Therefore, this approach uses the data contained in the sample to determine lower and upper confidence bounds for the population percentile.

Let's consider an example. Suppose we have the following random sample of size 20 from some population with an unknown distribution (displayed in Table 1). For convenience, the data are listed in ranked (ascending) order.

Table 1: Random Sample of Data (in Ascending Order)

Rank	1	2	3	4	5	6	7	8	9	10
Value	**0.49**	**0.59**	**0.86**	**1.01**	**1.24**	**1.25**	**1.81**	**2.01**	**2.29**	**2.66**
Rank	11	12	13	14	15	16	17	18	19	20
Value	**2.82**	**2.85**	**3**	**3.27**	**4.44**	**5.14**	**5.53**	**5.6**	**6.06**	**6.29**

What is a 95% CI for the median and the 75th percentile? Using statistical software, we can estimate the median and 75th percentile and their respective CIs. To perform this analysis in JMP (V.12), with your data opened in a data table, select "Distribution" under the "Analyze" menu. Select your variable of interest in the "y" box, and click OK. In the results window, go to the red triangle, select display options, and then select custom quantiles (Figure 1). Enter in the percentiles of interest (0.50 for median, 0.25 for 25th percentile, 0.75 for 75th percentile, etc.) [see Figure 2]. The results are now displayed in the distribution results window (Figure 3). JMP displays the point estimate for the median as well as the lower and upper confidence limits. JMP also displays the actual confidence. As explained in the Appendix, the actual confidence may not be equal to the desired confidence because the approach uses the Binomial distribution (a discrete distribution) to determine which values in the sample are the lower and upper confidence limits. Particularly when the sample size is small, the CIs may have a much smaller level of confidence than desired.

As seen in Figure 3, the estimate of the median is $\hat{x}_{0.50} = 2.74$. Note that this is equal to $\left(X_{(10)} + X_{(11)}\right)/2 = (2.66 + 2.82)/2$ from Table 1. The JMP results show that the 95% CI for the median is (1.25, 2.44) and the actual coverage is just above 95%. The estimate of the 75th percentile is 4.965 with an approximate 95% CI of (2.85, 6.29) which correspond to $X_{(11)}$ and $X_{(20)}$. The actual coverage for this CI is also just above 95%.

Now suppose we wish to find a 95% CI for the 95th percentile of the population based on the sample in Table 1. Figure 4 displays the JMP results for this scenario. The 95th percentile is estimated as 6.2785. The "95%" CI is (0.49, 6.29), which is the entire range of the sample data. Note that the actual coverage is just 64.15%, much lower than the desired 95% confidence. Because this dataset is so small, using this approach does not yield a CI with the desired confidence level. Suppose we took a sample of size 100 from the same population as the previous sample. The distribution analysis results from JMP are shown in Figure 5. First note that this data is clearly not normally distributed. The 95% CIs for the median, 75th, and 95th percentiles for this larger sample are more realistic and each have actual confidence slightly larger than the desired confidence 95% (see Figure 5).

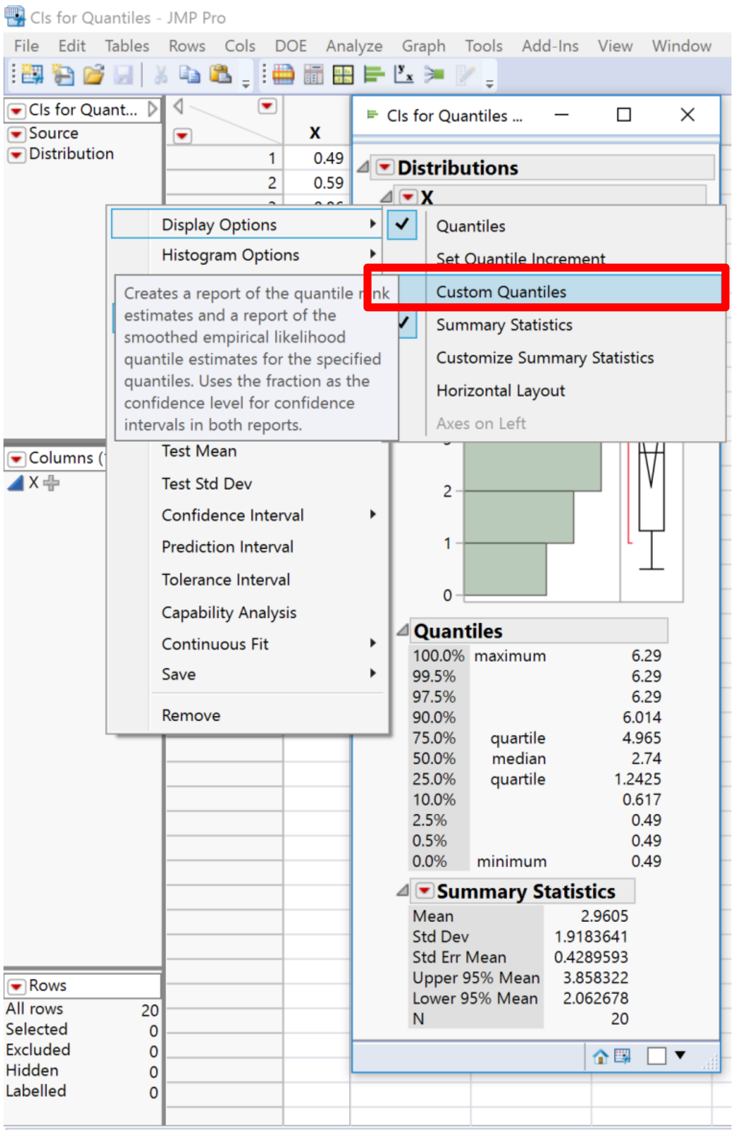


Figure 1: JMP Instructions Step 1

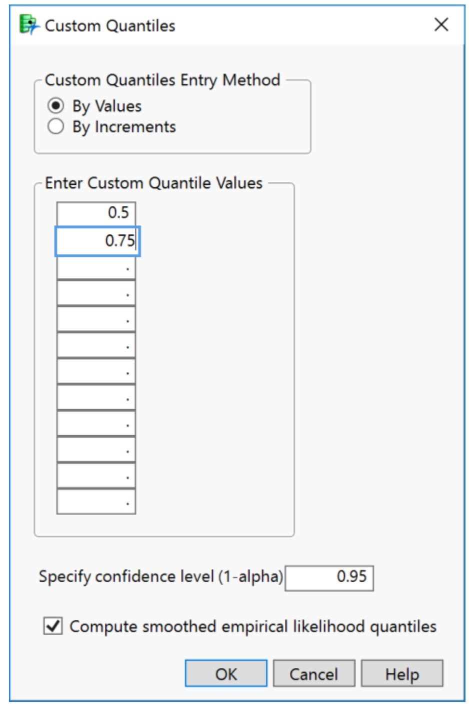
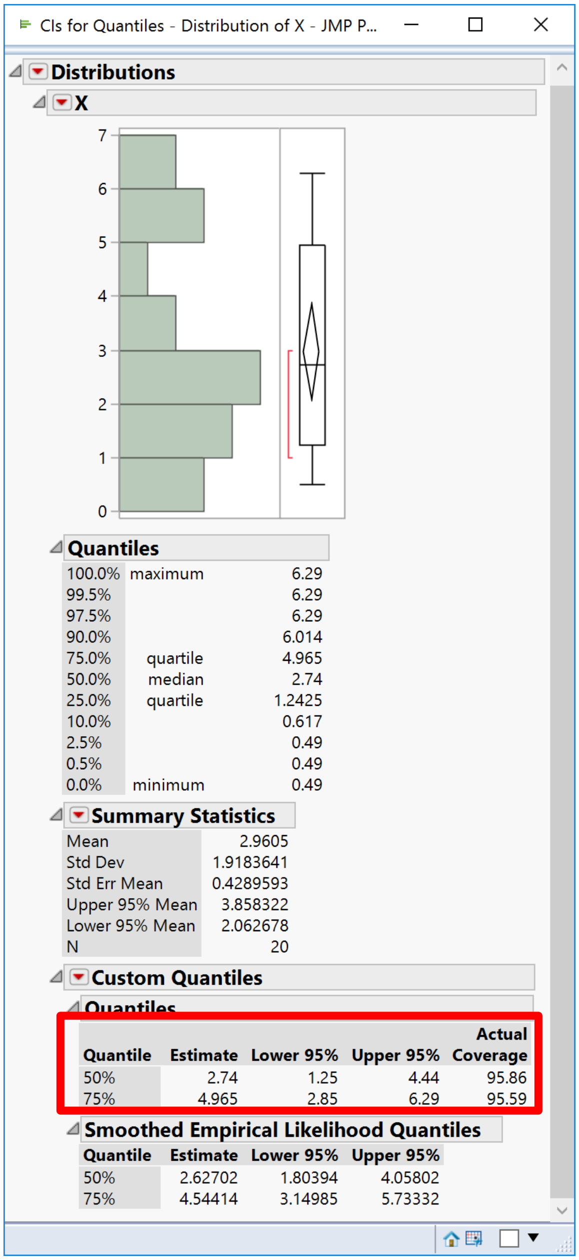


Figure 2: JMP Instructions Step



CIs for Quantiles - Distribution of X - JMP P...

Distributions

X

Quantiles

100.0%	maximum	6.29
99.5%		6.29
97.5%		6.29
90.0%		6.014
75.0%	quartile	4.965
50.0%	median	2.74
25.0%	quartile	1.2425
10.0%		0.617
2.5%		0.49
0.5%		0.49
0.0%	minimum	0.49

Summary Statistics

Mean	2.9605
Std Dev	1.9183641
Std Err Mean	0.4289593
Upper 95% Mean	3.858322
Lower 95% Mean	2.062678
N	20

Custom Quantiles

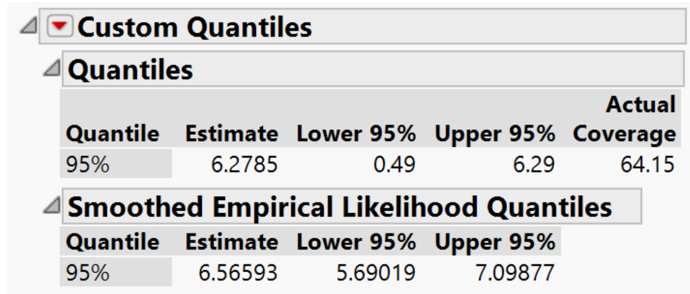
Quantiles

Quantile	Estimate	Lower 95%	Upper 95%	Actual Coverage
50%	2.74	1.25	4.44	95.86
75%	4.965	2.85	6.29	95.59

Smoothed Empirical Likelihood Quantiles

Quantile	Estimate	Lower 95%	Upper 95%
50%	2.62702	1.80394	4.05802
75%	4.54414	3.14985	5.73332

Figure 3: JMP Output



Custom Quantiles

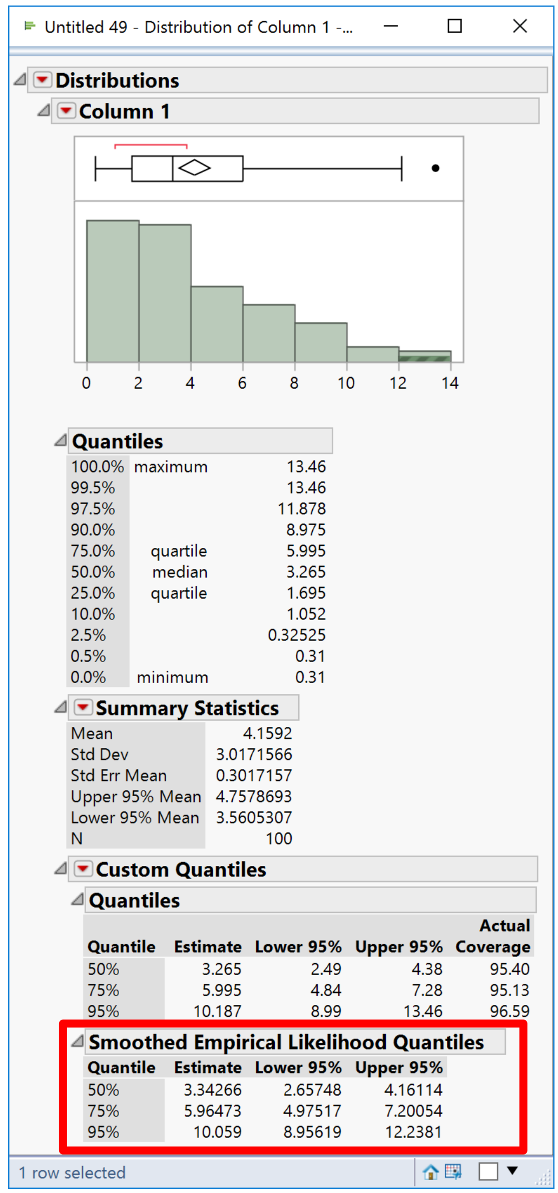
Quantiles

Quantile	Estimate	Lower 95%	Upper 95%	Actual Coverage
95%	6.2785	0.49	6.29	64.15

Smoothed Empirical Likelihood Quantiles

Quantile	Estimate	Lower 95%	Upper 95%
95%	6.56593	5.69019	7.09877

Figure 4: JMP results for 95th percentile



Untitled 49 - Distribution of Column 1 -...

Distributions

Column 1

Quantiles

100.0%	maximum	13.46
99.5%		13.46
97.5%		11.878
90.0%		8.975
75.0%	quartile	5.995
50.0%	median	3.265
25.0%	quartile	1.695
10.0%		1.052
2.5%		0.32525
0.5%		0.31
0.0%	minimum	0.31

Summary Statistics

Mean	4.1592
Std Dev	3.0171566
Std Err Mean	0.3017157
Upper 95% Mean	4.7578693
Lower 95% Mean	3.5605307
N	100

Custom Quantiles

Quantiles

Quantile	Estimate	Lower 95%	Upper 95%	Actual Coverage
50%	3.265	2.49	4.38	95.40
75%	5.995	4.84	7.28	95.13
95%	10.187	8.99	13.46	96.59

Smoothed Empirical Likelihood Quantiles

Quantile	Estimate	Lower 95%	Upper 95%
50%	3.34266	2.65748	4.16114
75%	5.96473	4.97517	7.20054
95%	10.059	8.95619	12.2381

1 row selected

Figure 5: JMP results for sample of size n = 100

Alternate Approaches

The mathematical details to determine the CIs for percentiles based on the distribution-free method described above is explained in the Appendix. JMP also calculates “Smoothed Empirical Likelihood Estimates” which is based on the work of Chen and Hall (1993). These results can be seen in Figure 3 and Figure 5. This is a more advanced method to calculate CIs for percentiles that uses a distribution constructed from the observed sample data. The method discussed previously was truly distribution-free and only required determining which ranked values in the sample to use as the lower and upper confidence bounds.

An alternate approach to finding CIs for percentiles (and any statistic) without relying on the distribution of the population is to use bootstrapping. In short, bootstrapping is a resampling method to estimate the sampling distribution of a statistic. The sampling distribution of the sample mean can be approximated by the Central Limit Theorem. The sampling distributions of other statistics, however, are often unknown (like with the median or other percentiles). To construct CIs on a statistic, we use properties of the sampling distribution to determine the confidence bounds. When this distribution is unknown, bootstrapping can estimate this sampling distribution which we can then use to construct the CIs. Bootstrapping will be discussed in a separate Best Practice. See Givens and Hoeting (2013) for details on bootstrapping.

Conclusion

It is possible to calculate CIs for the median and other percentiles. A word of caution worth reiterating: for small sample sizes, the method described here is not an ideal approach because of its limitations. With small sample sizes, we are not guaranteed to get a CI with the desired confidence level, particularly with the extreme percentiles (for example, 5% or 95% percentiles). It should also be noted that if the assumptions for a CI for the mean are valid for your sample, the CI for the mean will be more powerful than the method described here. When the assumptions are not valid however, or a percentile is the population characteristic of interest, we can accompany the point estimate with a CI. This will give us a realistic range of values for the population percentile of interest.

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Appendix

Here we explain the derivation of the confidence limits for percentiles. Note that there are two possible outcomes for each sample value X_i: it is either below the 100pth percentile or it's not (a binary outcome). The probability that a value falls below the 100pth percentile is p. Our sample size is fixed at n. These conditions (along with our random sample assumption) gives us the conditions to apply the Binomial distribution to determine the lower and upper confidence limits. The binomial distribution is a common distribution for a discrete random variable and, for example, can be used to estimate the number of successes (or failures) in n trials. Therefore, a 100(1-α)% CI that the 100pth percentile will fall between the jth and kth order statistic $X_{(j)}$ and $X_{(k)}$ is (http://www.milefoot.com/math/stat/ci-medians.htm):

$$\mathrm{P}\left(X_{(j)} \leq x_p \leq X_{(k-1)}\right) = \sum_{i=j}^{k-1} \frac{n!}{(n-i)!\,i!} p^i (1-p)^{n-i} \approx 1-\alpha$$

Consider the sample data in Table 1 where we wanted to determine a 95% CI of the median. Table 2 shows the probabilities for the binomial distribution for the median and the given sample size (n = 20, p = 0.50). This table supplies the probabilities that the percentile falls in the i^{th} subinterval of the ranked data. For example, $i = 0$ corresponds to the case where the pth population percentile falls below the minimum in the sample, $i = 1$ corresponds to the case where the percentile falls between the first and second order statistics, and $i = n$ corresponds to the case where the percentile is greater than the maximum (see Figure 6 for a graphical representation of this up to $i = 5$).

Order statistic:	____	$X_{(1)}$	____	$X_{(2)}$	____	$X_{(3)}$	____	$X_{(4)}$	____	$X_{(5)}$	____
i^{th} subinterval:	0		1		2		3		4		5

Figure 1. Graphical Representation of Table 2

We want to find values $X_{(j)}$ and $X_{(k-1)}$ such that $\mathrm{P}\left(X_{(j)} \leq x_{0.50} \leq X_{(k-1)}\right) \approx 1-\alpha$. The probabilities in Table 2 are calculated from the binomial distribution such that:

$$P(X=i)=\frac{n!}{(n-i)!\,i!}p^i(1-p)^{n-i}$$

Table 2. Binomial Probabilities for Median n = 20, p = 0.50

$X=i$	$P(X=i)$	$X=i$	$P(X=i)$
0	0	11	0.16018
1	0.00002	12	0.12013
2	0.00018	13	0.07393
3	0.00109	14	0.03696
4	0.00462	15	0.01479
5	0.01479	16	0.00462
6	0.03696	17	0.00109
7	0.07393	18	0.00018
8	0.12013	19	0.00002
9	0.16018	20	0.00000
10	0.1762		

Table 3 sorts these probabilities from largest to smallest to identify the set of subintervals with the desired confidence.

Table 3. Binomial Probabilities for Median n = 20, p= 0.50 (Sorted Descending)

$X=i$	$P(X=i)$	$X=i$	$P(X=i)$
10	**0.1762**	16	0.00462
11	**0.16018**	4	0.00462
9	**0.16018**	17	0.00109
12	**0.12013**	3	0.00109
8	**0.12013**	18	0.00018
13	**0.07393**	2	0.00018
7	**0.07393**	19	0.00002
14	**0.03696**	1	0.00002
6	**0.03696**	20	0.00000
15	0.01479		

Using Table 3, therefore, we can say:

$$\mathrm{P}\left(X_{(6)} \leq x_p \leq X_{(14)}\right) = \sum_{i=6}^{14} \frac{n!}{(n-i)!\,i!} p^i (1-p)^{n-i}$$
$$= 0.03696 + 0.07393 + 0.12013 + 0.16018 + 0.1762 + 0.16018 + 0.12013 + 0.07393 + 0.03696$$
$$= 0.9586$$

The confidence bounds for the 95% CI begin at the 6th subinterval ($X_{(6)}$) and end at the end of the 14th subinterval ($X_{(15)}$). This yields a 95% (actually 95.86%) CI for the median of $\left(X_{(6)}, X_{(15)}\right) = (1.25, 4.44)$ by referring to the ranked values in **Table 1**. Note that this matches the output from JMP in Figure 3. Note also that because of the discrete nature of the binomial distribution, we may not be able to get a CI with confidence exactly equal to $1 - \alpha$. And as discussed in the main text, for small sample sizes, the actual confidence can be much lower than the desired confidence.